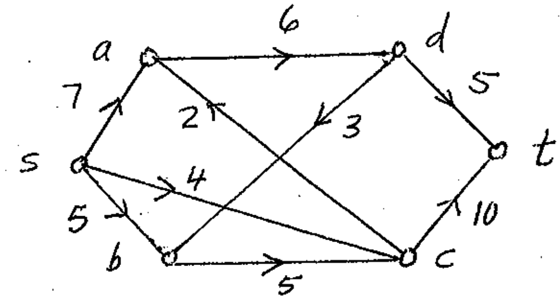
MAD 3305 - GRAPH THEORY
TEST #2 - SPRING 2015

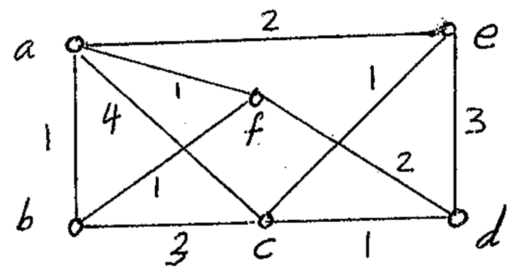
FLORIDA INT'L UNIV.
TIME: 75 min.

Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

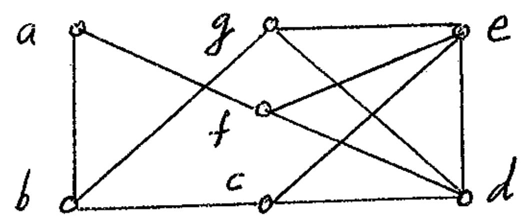
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

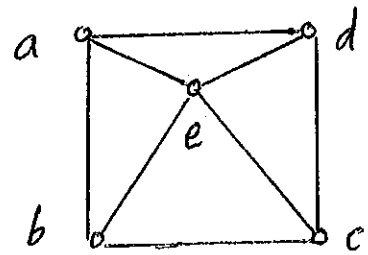


(16) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(24) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm.*

(b) If G is graph with no odd cycles, prove that $\chi(G) \leq 2$.



(15) 5(a) Define what is a *maximal matching* in a graph G.

(b) Use *Ore's Theorem* to prove that any graph G with $\deg(x) + \deg(y) \geq p-1$, for all pairs of non-adjacent vertices x &y, has a *Hamilton path*. Here $p = |V(G)|$.

(15) 6(a) Define what is a *simple polyhedron* and what is a *polyhedral graph.*

(b) Let G be a *simple polyhedron* with p vertices & q edges and no triangular or rectangular faces. Prove that $3q \leq 5p - 10$.

[*You may use any theorem proved in class for Qu.#6*]

MAD 3305 - Graph Theory — Florida Int'l University

Solutions to Test #2 — Spring 2015

1. 1st aug. semi-path:

$s \xrightarrow{(0,4)} c \xrightarrow{(0,10)} t$

Slacks: 4 10 $\mu_1 = 4$

2nd aug. semi-path:

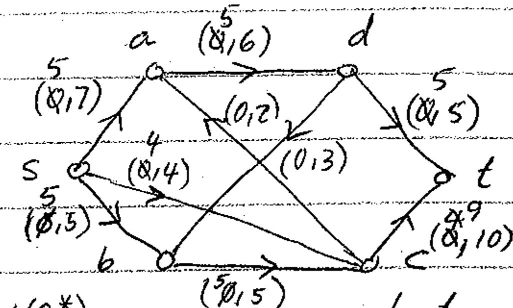
$s \xrightarrow{(0,7)} a \xrightarrow{(0,6)} d \xrightarrow{(0,5)} t$

Slacks: 7 6 5 $\mu_2 = 5$

3rd aug. semi-path

$s \xrightarrow{(0,5)} b \xrightarrow{(0,5)} c \xrightarrow{(4,10)} t$

Slacks 5 5 6 $\mu_3 = 5$



$Val(f^*)$ = net flow into t

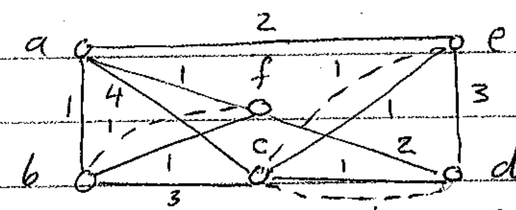
$= f^*(\vec{dt}) + f^*(\vec{ct}) = 5 + 9 = 14.$

$S^* = \{u \in V;$ There is an aug. semi-path from s to $u\} = \{s, a, d, b\}$

$c(S^*)$ = sum of outward capacities $= c(\vec{sc}) + c(\vec{bc}) + c(\vec{dt}) = 4+5+5 = 14.$

2. Odd vertices: b, d, e, f

	b	d	e	f
b	.	3	3	1
d		.	2	2
e			.	3
f				.

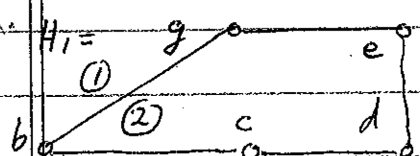


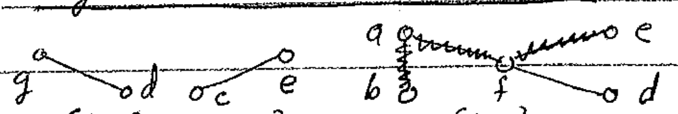
$\{b,d\} + \{e,f\}$ $3 + 3 = 6$ $\{b,e\} + \{d,f\}$ $3 + 2 = 5$ $\{b,f\} + \{d,e\}$ $1 + 2 = 3$

A Min. postman walk is: $b \overset{1}{—} f \overset{1}{—} b \overset{3}{—} c \overset{4}{—} a \overset{1}{—} f \overset{2}{—} d \overset{1}{—} c$ $\overset{1}{—} d \overset{3}{—} e \overset{1}{—} c \overset{1}{—} e \overset{2}{—} a \overset{1}{—} b.$ Total length = 22.

3. Segments of G relative to H_i

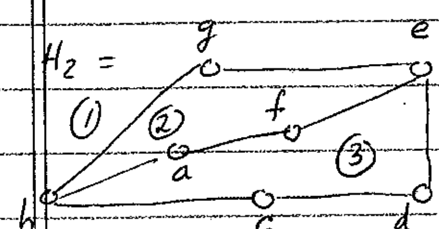
$H_1 =$

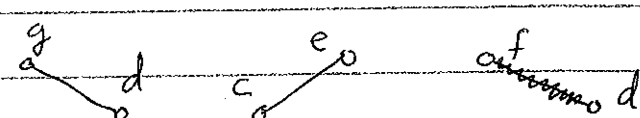




$R_1(S)$: $\{1,2\}$ $\{1,2\}$ $\{1,2\}$ ↑ R

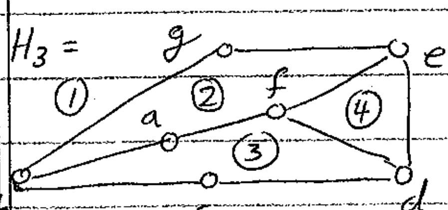
$H_2 =$

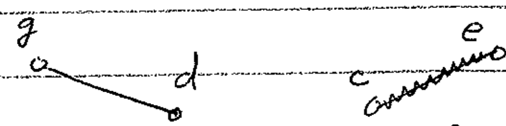




$R_2(S)$: $\{1\}$ $\{1,3\}$ $\{3\}$ ↑ R

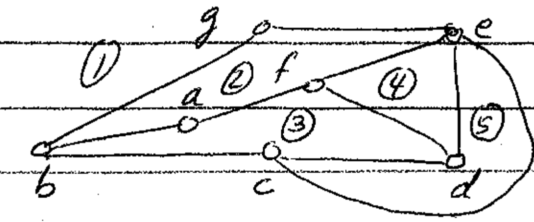
$H_3 =$



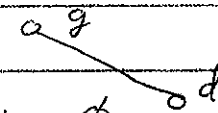


$R_3(S)$: $\{1\}$ $\{1\}$ ↑ R

3. $H_4 =$

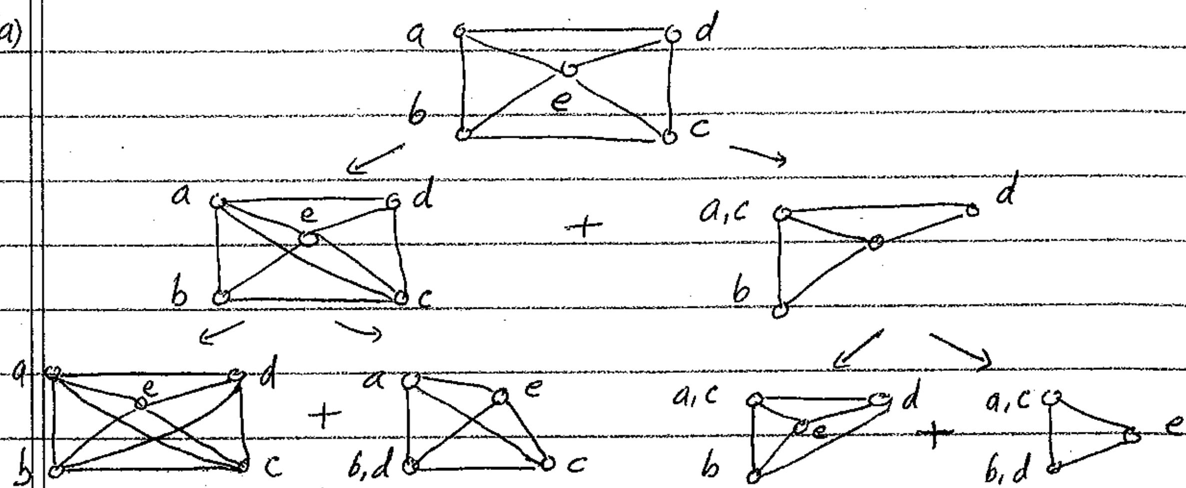


$\mathcal{R}_4(S)$: ϕ



G is NON-PLANAR

4(a)



$$P_G(\lambda) = P_{K_5}(\lambda) + 2P_{K_4}(\lambda) + P_{K_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) +$$
$$2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$$

(b) Let $F = T_1 \cup T_2 \cup \dots \cup T_n$ be a spanning forest of G. For each tree T_i in F, select a vertex v_i as the root and so make each T_i into a rooted tree. Color the vertices in the even levels of T_i with color #1 & the vertices in the odd levels with color #2. Now add back the other edges of G from $E(G) - E(F)$, one at a time. Each time we add an edge we must connect two vertices with different colors (otherwise we would get an odd cycle in G - which is impossible). Hence our coloring of the vertices of G will be a legal coloring. So $\chi(G) \leq 2$.

5(a) A matching in a graph G is any set of edges $M \subseteq E(G)$ such that no two edges in M share a common endpoint. A maximal matching in G is any matching M' in G such that M' is not a proper subset of any other matching in G.

(b) Let G' be the graph obtained by adding a new vertex v_{p+1} and edges from v_{p+1} to each of the vertices of G. Then $|V(G')| = p+1$.

5(b) Now for any pair of non-adjacent vertices x & y in G' we will have $\deg_{G'}(x) + \deg_{G'}(y) = \{\deg_G(x)+1\} + \{\deg_G(y)+1\}$

$= \{\deg_G(x) + \deg_G(y)\} + 2 \geq (p-1) + 2 = p+1 = |V(G')|$.

So by Ore's Theorem, G' will have a Hamilton cycle C. Now if we remove the vertex v_{p+1} from the cycle C, we will get a Hamilton path in G. Thus G will always have a Hamilton path if $\deg(x) + \deg(y) \geq |V(G)| - 1$ for each pair of non-adj. vertices in G.

6(a) A *simple polyhedron* is a solid figure which is bounded by plane polygonal faces and which can be continuously deformed into a solid sphere. A *polyhedral graph* is any graph which can be obtained by considering the vertices & edges of a simple polyhedron as vertices & edges of a graph

(b) First observe that since G is a polyhedral graph, G will be a connected planar graph by a theorem proved in class. Let r = no. of faces of G. Then each of the faces F_i of G will be bounded by at least 5 edges. So

$$5r = 5 + 5 + 5 + \cdots + 5 \quad (r \text{ times})$$

$$\leq e(F_1) + e(F_2) + \cdots + e(F_r), \text{ where } e(F_i) = \text{no. of edges in the face } F_i$$

$$= 2q \quad \text{because each edge is in 2 faces.}$$

Hence $5r \leq 2q$. But $r = q + 2 - p$ by Euler's Planarity formula for connected planar graphs. Thus

$$5(q+2-p) \leq 2q. \quad \therefore \quad 5q + 10 - 5p \leq 2q$$

So $3q \leq 5p - 10$ and we are done. END