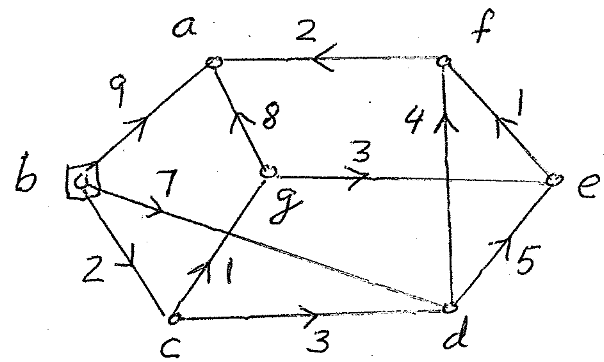
MAD 3305 - GRAPH THEORY
TEST #1 - SPRING 2016

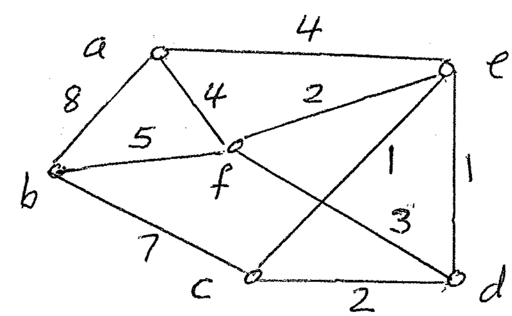
FLORIDA INT'L UNIV.
TIME: 75 min.

Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the *distances* from b to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2(a) Find a *graph* with degree sequence $\langle 3, 3, 3, 3, 2\rangle$ by using the *Graphical Sequence Algorithm.*
(b) For the graph on the right, find a *minimal spanning tree* by using *Kruskal's Algorithm.*



(20) 3(a) Find the *tree* corresponding to the sequence $\langle 1, 6, 4, 4\rangle$ via *Prufer's Tree Decoding Algorithm.*
(b) The five characters *a, b, c, d, e* occur with frequencies *2, 2, 3, 7, 8* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm*.

(15) 4(a) Define what is the *adjacency matrix* A_G of a graph with $V(G)=\{1,2,3, \dots, p\}$.
(b) Prove that in any tree T, we always have $|E(T)| = |V(T)| - 1$.

(15) 5(a) Define what is the *distance from u to v* in a weighted *digraph* G.
(b) Let G be a graph such that $deg(x) + deg(y) \geq p-1$ for any pair of non-adjacent vertices x and y. Prove that G must be a connected graph.

(15) 6(a) Define what is a *source-separating set U of vertices* in a network $N = \langle G,s,t,c\rangle$ and what is the *capacity of the cut determined* by U.
(b) Prove that in any 3-ary tree T we will always have $h(T) \geq \log_3(2p+1) - 1$.
Here $p = |V(T)|$

MAD 3305 - Graph Theory — Florida International Univ.

Solutions to Test #1 — Spring 2016

1.

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	v_0
∞	$\underline{0}$	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	b
9	·	$\underline{2}$	7	∞	∞	∞	$\{a,c,d,e,f,g\}$	1	c
9	·	·	5	∞	∞	$\underline{3}$	$\{a,d,e,f,g\}$	2	g
9	·	·	$\underline{5}$	6	∞	·	$\{a,d,e,f\}$	3	d
9	·	·	·	$\underline{6}$	9	·	$\{a,e,f\}$	4	e
9	·	·	·	·	$\underline{7}$	·	$\{a,f\}$	5	f
$\underline{9}$	·	·	·	·	·	·	$\{a\}$	6	a
·	·	·	·	·	·	·	$\emptyset$ STOP		
$d(b,\cdot) = 9$	0	2	5	6	7	3			

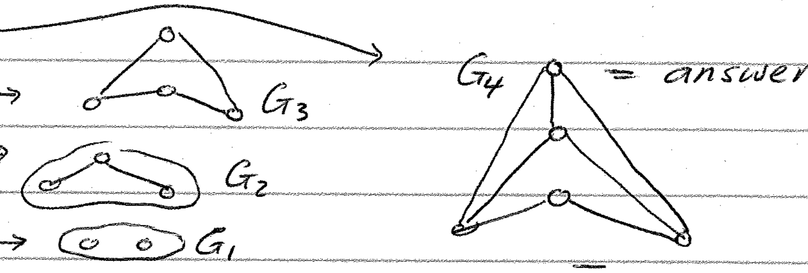
2(a) 3, 3, 3, 3, 2

2, 2, 2, 2 →

1, 1, 2 →

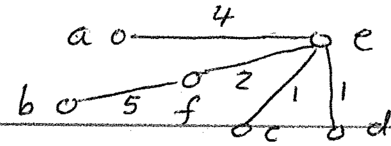
2, 1, 1

0, 0 →



(b) $e_1 = \overline{de}$, $e_2 = \overline{ce}$, $e_3 = \overline{cd}$, $e_4 = \overline{ef}$, $e_5 = \overline{df}$, $e_6 = \overline{ae}$, $e_7 = \overline{af}$, $e_8 = \overline{bf}$, $e_9 = \overline{bc}$, $e_{10} = \overline{ab}$. (Check – yes, G has 10 edges.)

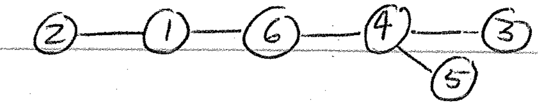
$E(T)$	Parts of the Partition P	i	endpoints(e_i)
$\emptyset$	$\{a\}\{b\}\{c\}\{d\}\{e\}\{f\}$	1	$\{d,e\}$
$\{\overline{de}\}$	$\{a\}\{b\}\{c\}\{d,e\}\{f\}$	2	$\{c,e\}$
$\{\overline{de},\overline{ce}\}$	$\{a\}\{b\}\{c,d,e\}\{f\}$	3	$\{c,d\}$
don't add $\overline{cd}$	unchanged	4	$\{e,f\}$
$\{\overline{de},\overline{ce},\overline{ef}\}$	$\{a\}\{b\}\{c,d,e,f\}$	5	$\{d,f\}$
don't add $\overline{df}$	unchanged	6	$\{a,e\}$
$\{\overline{de},\overline{ce},\overline{af},\overline{ae}\}$	$\{a,c,d,e,f\}\{b\}$	7	$\{a,f\}$
don't add $\overline{af}$	unchanged	8	$\{b,f\}$
$\{\overline{de},\overline{ce},\overline{af},\overline{ae},\overline{bf}\}$	$\{a,b,c,d,e,f\} = V(G)$, STOP		

2(b) Minimal spanning Tree = 

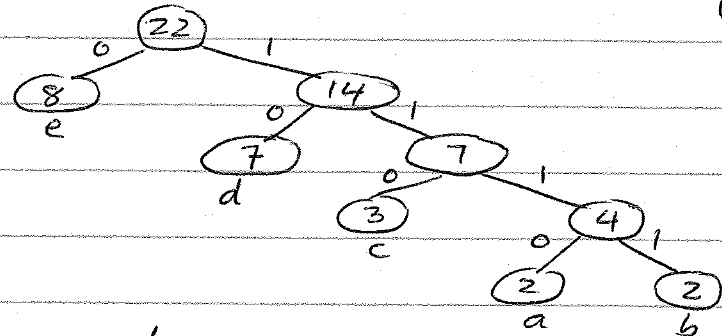
$p = |S| + 2 = 4 + 2 = 6$

3(a)

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$l(i)$ — $s(i)$
2	1	1	3	1	2	1	2 — 1
1	0	1	3	1	2	2	1 — 6
0	.	1	3	1	1	3	3 — 4
.	.	0	2	1	1	4	5 — 4
.	.	.	1	0	1	5	4 — 6

T = 

(b)
a b c d e
2, 2, 3, 7, 8
3, 4, 7, 8
7, 7, 8
8, 14
22



Char.	a	b	c	d	e
Code	1110	1111	110	10	0
$f(c_i)$	2	2	3	7	8
$l(c_i)$	4	4	3	2	1

WPL (coding) $= 2(4) + 2(4) + 3(3) + 7(2) + 8(1) = 47.$

4(a) $A_G[i,j]$ = number of edges from i to j

(b) We shall prove the result by induction on $p = |V(T)|$.

<u>Basis</u>: If $p=1$, then $T \cong K_1$, so $|E(T)| = 0 = 1-1 = |V(T)|-1$. So the is true for $p=1$.

<u>Ind. Step</u>: Suppose the result is true for all trees with $\leq p$ vertices. Let T be any tree with $p+1$ vertices and e be any edge in T. Then $T - \{e\}$ = a disjoint union of two trees T_1 and T_2. So $|E(T)| = |E(T_1)| + |E(T_2)| + |\{e\}| = (|V(T_1)|-1) + (|V(T_2)|-1) + 1 = |V(T)|-1$
So if the result is true for all T with $\leq p$ vertices, it will be true for all trees with $p+1$ vertices. <u>Conclusion</u>: By the strong Principle of Math Induction, it follows that the result is true for all trees.

5(a) $d(u,v) = \begin{cases} \text{length of the shortest directed path from } u \text{ to } v \text{ in } G \\ +\infty, \text{ if there is no directed path from } u \text{ to } v \text{ in } G. \end{cases}$

(b) Suppose that $\deg(x) + \deg(y) \geq p-1$ for any pair of non-adjacent vertices x & y in G. Now assume that G is a disconnected graph. Then we can find two vertices x_0 and y_0 such that there is no path from x_0 to y_0 in G. Let A = set of vertices in G that are adjacent to x_0 and B = set of vertices in G that are adjacent to y_0. Then $A \cap B = \emptyset$ because x_0 cannot be adjacent y_0 and $A \cup B \subseteq V(G) - \{x_0, y_0\}$. So $|A \cup B| \leq p-2$. Since $A \cap B = \emptyset$, $\deg(x_0) + \deg(y_0) = |A| + |B| = |A \cup B| \leq p-2$. But this contradicts the fact that $\deg(x) + \deg(y) \geq p-1$ for any pair of non-adjacent vertices in G. Hence our assumption must be wrong. So G must be a connected graph.

6 (a) A source-separating set, U, of vertices in N is any subset $U \subseteq V(G)$ with $s \in U$ and $t \notin U$. $c[Cut(U)] = \sum_{e \in Out(U)} c(e)$ where $Out(U) = \{e \in E(G) : e \text{ is a directed edge from } U \text{ to } \bar{U}\}$.

(b) Let T be any 3-ary tree and $k = h(T)$. Then T can have at most 3^0 vertices at level 0, at most 3^1 vertices at level 1, at most 3^2 vertices at level 2, ..., and at most 3^k vertices at level k. So total number of vertices

$$p \leq 3^0 + 3^1 + 3^2 + \cdots + 3^k = (3^{k+1} - 1)/(3-1)$$

$\therefore \quad 2p \leq 3^{k+1} - 1$. So $\quad 3^{k+1} \geq 2p+1$

Hence $\log_3(3^{k+1}) \geq \log_3(2p+1)$.

$\therefore \quad k+1 \geq \log_3(2p+1)$. Thus $k \geq \log_3(2p+1) - 1$.

Hence $h(T) \geq \log_3(2p+1) - 1$.

END.

Note: $\bar{U} = V(G) - U$, in part (a).