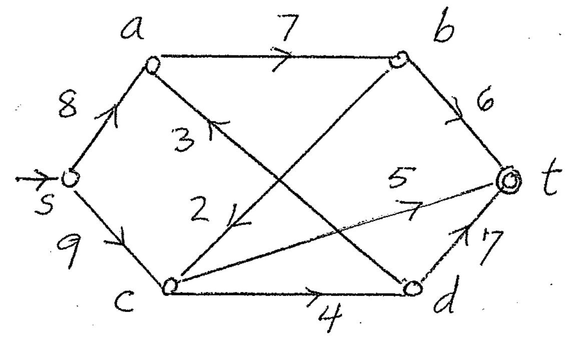
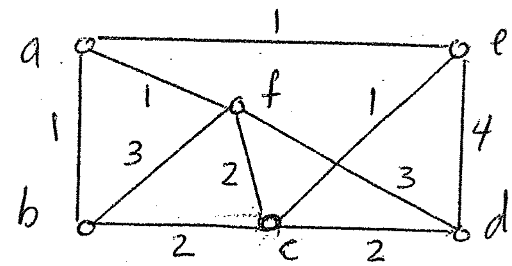
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.
TEST #2 - SPRING 2016 TIME: 75 min.

Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

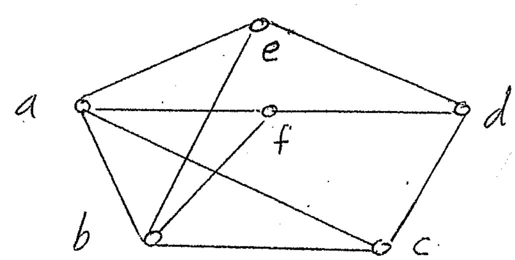
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm.* Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

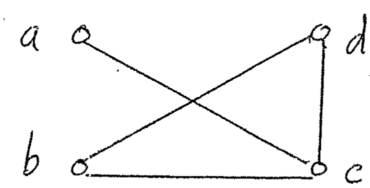


(18) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm.* [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm.*

(b) Prove $P_T(\lambda) = \lambda.(\lambda-1)^{n-1}$ for any tree T.



(15) 5(a) Define what is a *Hamilton-connected* graph *G*.

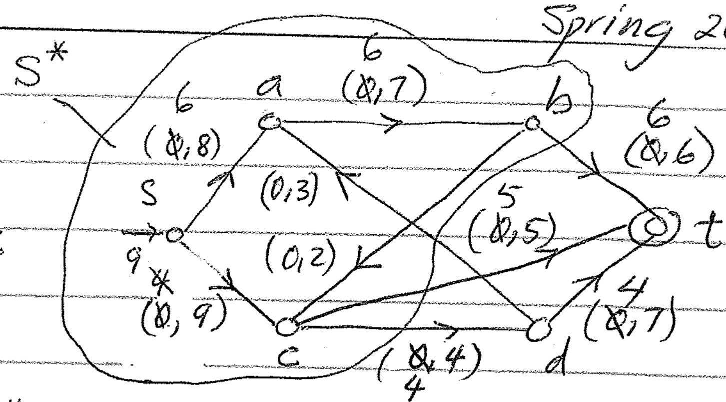
(b) Prove that in any planar graph with k components we have $r = q + (k+1) - p$. [*You may use Euler's formula for connected planar graphs, if needed.*]

(15) 6(a) Define what is the *dual of* planar graph *G* w.r.t. the planar embedding $\mathcal{E}$.

(b) Let G be a *self-dual connected* planar graph with p vertices and q edges. Prove that $q = 2p - 2$ and use this to find a self-dual planar graph with 5 vertices. [*You may use Euler's formula for connected planar graphs, if needed.*]

MAD 3305 – Graph Theory Florida International Univ.

Solutions to Test #2 Spring 2016



1(a) 1st Aug. semi-path:

$s \xrightarrow{(0,8)} a \xrightarrow{(0,7)} b \xrightarrow{(0,6)} t$

slacks: 8 7 6 $\mu_1 = 6$

2nd Aug. semi-path:

$s \xrightarrow{(0,9)} c \xrightarrow{(0,4)} d \xrightarrow{(0,7)} t$

slacks: 9 4 7 $\mu_2 = 4$

3rd Aug. semi-path

$s \xrightarrow{(4,9)} c \xrightarrow{(0,5)} t$

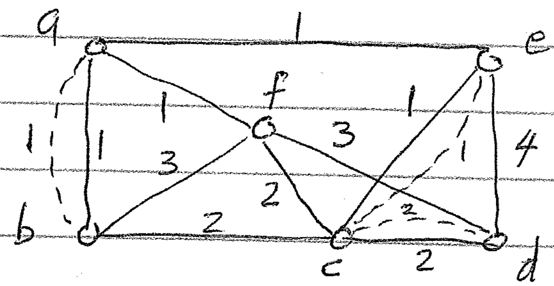
slacks 5 5 $\mu_3 = 5$

$Val(f^*)$ = net flow into t
$= 6+5+4 = 15$

$c(S^*)$ = sum of outward capacities
$= 6+5+4 = 15$ ✓

(b) $S^* = \{u \in V(G)$: there is an aug. semi-path from s to $u\} = \{s, a, b, c\}$

2(a) Odd vertices are: a, b, d, e.



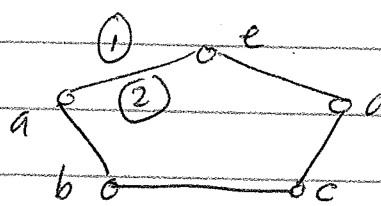
dist.	a	b	d	e
a	.	1	4	1
b		.	4	2
d			.	3
e				.

$\{a,b\}+\{d,e\}$ $1+3=4$ $\{a,d\}+\{b,e\}$ $4+2=6$ $\{a,e\}+\{b,e\}$ $1+4=5$

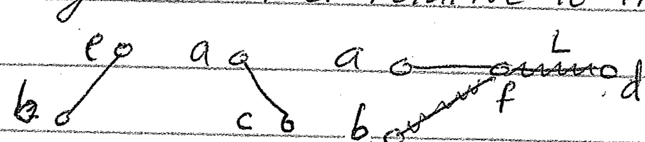
(b) Min postman walk: $a \overset{1}{—} b \overset{1}{—} a \overset{1}{—} f \overset{3}{—} b \overset{2}{—} c \overset{2}{—} f \overset{3}{—} d \overset{2}{—} c \overset{1}{—} e \overset{1}{—} c \overset{2}{—} d \overset{4}{—} e \overset{1}{—} a.$

Length of wak $= 20+4 = 24.$

3. $H_1 =$

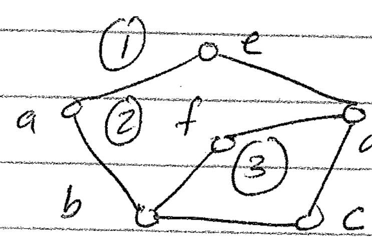


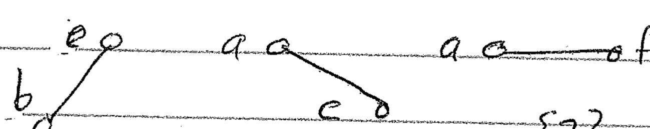
segments of G relative to H_i



$R_1(S) = \{1,2\}$ $\{1,2\}$ $\{1,2\}$

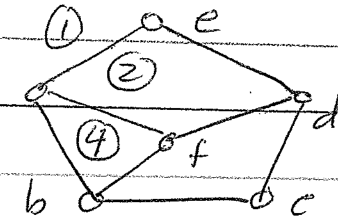
$H_2 =$

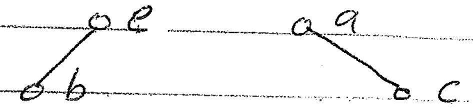




$R_2(S) = \{1,2\}$ $\{1\}$ $\{2\}$

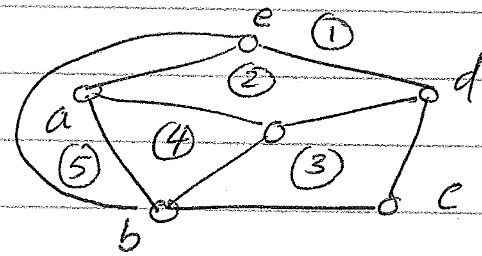
3. $H_3 =$

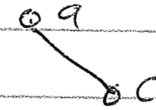




$R_3(S) = \{1\}$ $\qquad\qquad$ $\{1\}$

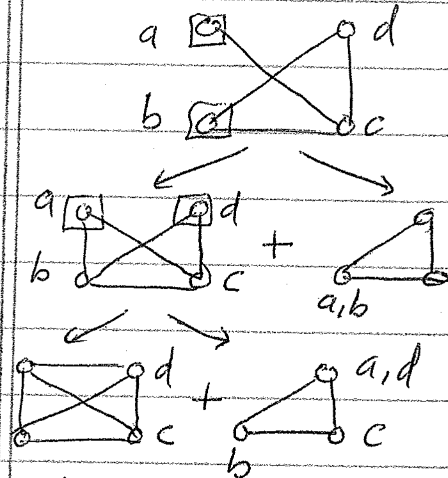
$H_4 =$





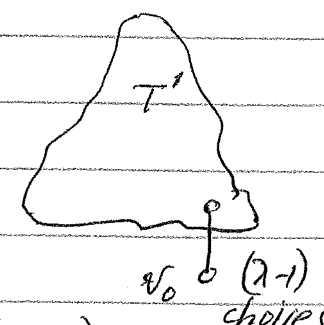
$R_4(S) = \emptyset$ - NON-PLANAR.

4 (a)



$$\begin{aligned}
P_G(\lambda) &= P_{K_4}(\lambda) + P_{K_3}(\lambda) + P_{K_3}(\lambda) \\
&= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) \\
&= \lambda(\lambda-1)(\lambda-2)\left[(\lambda-3)+2\right] \\
&= \lambda(\lambda-1)(\lambda-2)\left[\lambda-1\right] \\
&= \lambda(\lambda-1)^2(\lambda-2)
\end{aligned}$$

4 (b) We will prove the result by induction on $n = |V(T)|$. If $n=1$, then $T \cong K_1$. So $P_T = \lambda = \lambda(\lambda-1)^{1-1}$. So the result is true for all trees with 1 vertex. Now suppose the result is true for all trees with n vertices. Let T be a tree with $n+1$ vertices. Choose any leaf v_0 in T and put $T' = T - \{v_0\}$. (T must have a leaf because $n+1 \geq 2$ and the endpts of any maximal path in T will be leaves.) Then $P_T(\lambda) = P_{T'}(\lambda) \cdot (\lambda-1)$ $= \lambda(\lambda-1)^{n-1} \cdot (\lambda-1) = \lambda(\lambda-1)^{n+1-1}$. So if the result is true for all trees with n vertices, it will be true of all trees with $n+1$ vertices. By the Principle of Mathematical Induction, it follows that the result is true for all n and consequently for all trees.



5(a) A graph G is Hamilton connected if there is a Hamilton path between any two vertices of G.

(b) Let G be a planar graph with k connected components $G_1, \ldots, G_k$. Then $r(G_i) = q(G_i) + 2 - p(G_i)$ by Euler's Planarity formula for each $i = 1, \ldots, k$. So

$$\sum_{i=1}^{k} r(G_i) = \sum_{i=1}^{k} q(G_i) + \sum_{i=1}^{k} 2 - \sum_{i=1}^{k} p(G_i).$$

But the infinite region is counted k times in $\sum_{i=1}^{n} r(G_i)$ so $r(G) + (k-1) = \sum_{i=1}^{k} = q(G) + 2k - p(G)$. Thus

$$r(G) = q(G) + (k+1) - p(G).$$

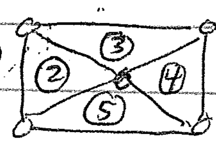
6(a) The dual $G^*_{\mathcal{E}}$ of a planar graph G w.r.t the embedding $\mathcal{E}$ is defined as follows. $V(G^*_{\mathcal{E}})$ = set of the regions into which $\mathcal{E}$ partitions the plane, and for each edge that is a boundary between the regions R_1 & R_2, we get an edge in $E(G^*_{\mathcal{E}})$.

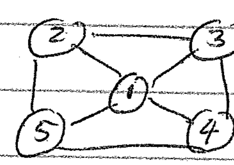
(b) Suppose G is a self-dual connected planar graph with p vertices & q edges. Let $\mathcal{E}$ be any planar embedding of G. Then $r(G) = q(G) + 2 - p(G)$ by Euler's planarity formula. Now since G is self-dual, $G^*_{\mathcal{E}} \cong G$. So $p(G^*_{\mathcal{E}}) = p(G)$. But $p(G^*) = r(G)$ because $V(G^*_{\mathcal{E}})$ = set of regions into which $\mathcal{E}$ partitions the plane. Hence $r(G) = p(G)$

$\therefore \quad p(G) = q(G) + 2 - p(G)$

$\therefore \quad p = q + 2 - p$ and so $q = 2p - 2$.

(c) If G is a self-dual graph with 5 vertices, then G must have $2(5) - 2 = 8$ edges. Now just look for a graph with 5 vertices & 8 edges which is self-dual.

Take $G =$ 

. Then $G^* =$ 

$\cong G$ and we are done.