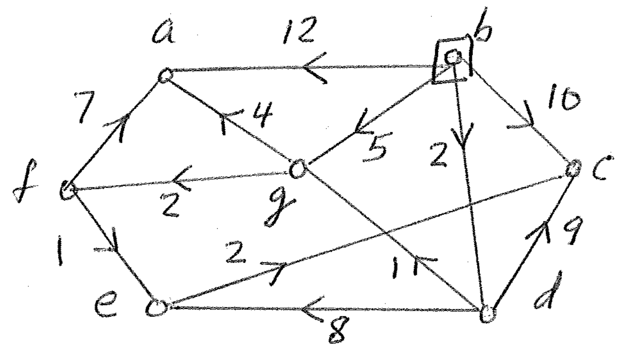
MAD 3305 - GRAPH THEORY
TEST #1 - SPRING 2017

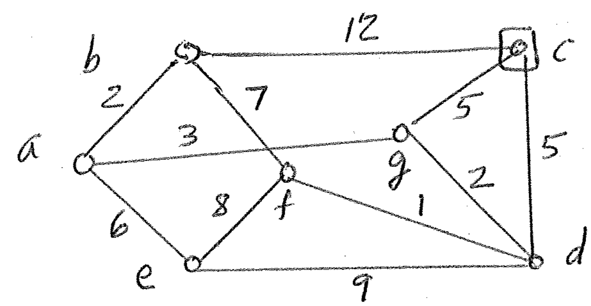
FLORIDA INT'L UNIV.
TIME: 75 min.

Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the *distances* from b to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2(a) Find a *graph* with degree sequence $\langle 4, 3, 3, 2, 2\rangle$ by using the *Graphical Sequence Algorithm.*
(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* and starting at *c*.



(20) 3(a) Find the *tree* that corresponds to the sequence $\langle 3, 2, 1, 3\rangle$ via *Prufer's Tree Decoding Algorithm.*
(b) The five characters *a, b, c, d, e* occur with frequencies *7, 2, 4, 5, 12* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4(a) Define what is the *adjacency matrix* of a digraph G with $V(G) = \{1,2,3,\ldots,p\}$.
(b) Prove that for any tree T, we will always have $|E(T)| = |V(T)| - 1$.

(15) 5(a) Define what is the *distance* $d(u,v)$ from u *to* v in a *weighted digraph G.*
(b) Let G be a graph with p vertices such that $deg(v) \geq (p-1)/2$ for each v in G. Prove that G must be a *connected* graph.

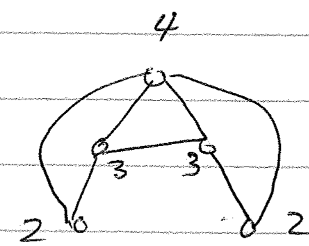
(15) 6(a) Define what is a *source-separating set U* of vertices in a network $N = \langle G,s,t,c\rangle$ and define what is the *capacity of the cut* determined by U.
(b) Prove that in any *non-trivial* graph G we can always find two vertices with the same degree.

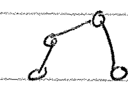
MAD 3305 - Graph Theory — Florida Int'l Univ.

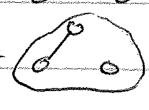
Solutions to Test #1 — Spring 2017

1.

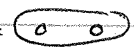
$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	x_0
∞	$\underline{0}$	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	b
12	.	10	$\underline{2}$	∞	∞	5	$\{a,c,d,e,f,g\}$	1	d
12	.	10	.	10	∞	$\underline{3}$	$\{a,c,e,f,g\}$	2	g
7	.	10	.	10	$\underline{5}$	.	$\{a,c,e,f\}$	3	f
7	.	10	.	$\underline{6}$	.	.	$\{a,c,e\}$	4	e
$\underline{7}$	.	8	.	.	.	.	$\{a,c\}$	5	a
.	.	$\underline{8}$	.	.	.	.	$\{c\}$	6	c
.	.	.	.	.	.	.	$\emptyset$		
7	0	8	2	6	5	3	$= d(b,\cdot)$		

2(a) $4, \underline{3,3,2,2}$ $\longrightarrow$ $G_4 =$ 

$2, \underline{2,1},1 \rightarrow G_3 =$ 

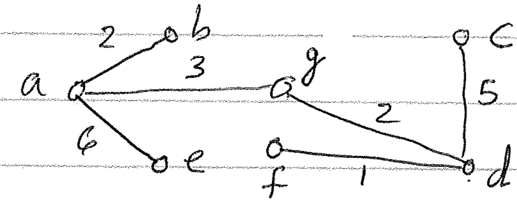
$1,0,1 \rightarrow$ $G_2 =$ 

reorder $\underline{1},1,0$

$0,0 \longrightarrow G_1 =$ 

(b)

a	b	c	d	e	f	g	$V(T)$	$E(T)$	i	x_0
∞	∞	$\underline{0}$	∞	∞	∞	∞	$\{c\}$	$\emptyset$	0	c
∞	12	.	$\underline{5}$	∞	∞	5	$\{c,d\}$	$\{\overline{cd}\}$	1	d
∞	12	.	.	9	$\underline{1}$	2	$\{c,d,f\}$	$\{\overline{cd},\overline{df}\}$	2	f
∞	7	.	.	8	.	$\underline{2}$	$\{c,d,f,g\}$	$\{\overline{cd},\overline{df},\overline{dg}\}$	3	g
$\underline{3}$	7	.	.	8	.	.	$\{a,c,d,f,g\}$	$\{\overline{cd},\overline{df},\overline{dg},\overline{ga}\}$	4	a
.	$\underline{2}$	.	.	6	.	.	$\{a,b,c,d,f,g\}$	$\{\overline{cd},\overline{df},\overline{dg},\overline{ga},\overline{ab}\}$	5	b
.	.	.	.	$\underline{6}$	.	.	$\{a,b,c,d,e,f,g\}$	$\{\overline{cd},\overline{df},\overline{dg},\overline{ga},\overline{ab},\overline{ae}\}$	6	e

$T_{min} =$ 

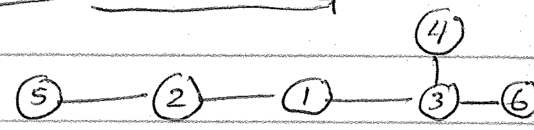
$w(T) = 5+1+2+3+2+6$

$= 19$ (not requested).

3(a) $\underset{\sim}{s} = \langle 3, 2, 1, 3 \rangle$ so $p = |\underset{\sim}{s}| + 2 = 4 + 2 = 6$

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	$\ell(i)$ — $s(i)$
2	2	$\underline{3}$	$\underline{1}$	1	1	4 — 3
2	$\underline{2}$	2	0	1	1	5 — 2
$\underline{2}$	$\underline{1}$	2	0	0	1	2 — 1
$\underline{1}$	0	$\underline{2}$	0	0	1	1 — 3
0	0	1	0	0	1	3 — 6

T =



(b)

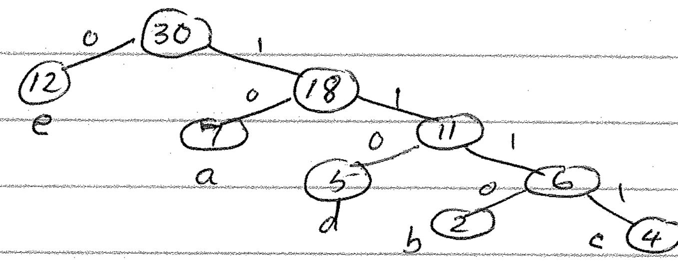
$\underbrace{2, 4}_{\to 6}, 5, 7, 12$

$\underbrace{5, 6}_{\to 11}, 7, 12$

$\underbrace{7, 11}_{\to 18}, 12$

$\underbrace{12, 18}_{\to 30}$

30



Character	a	b	c	d	e
Codes	10	1110	1111	110	0
Frequencies	7	2	4	5	12
Length, $\ell(c_i)$	2	4	4	3	1

WPL (coding)
$= 7(2) + 2(4) + 4(4) + 5(3) + 12(1)$
$= 14 + 8 + 16 + 15 + 12 = 65.$

4 (a) The adjacency matrix A of a digraph with vertices $\{1, 2, \dots, p\}$ is the $p \times p$ matrix defined by $A[i,j] =$ no. of dir. edges from i to j.

(b) We will prove the result by strong induction on $p = |V(T)|$.

Basis: If $p = 1$, then $T \cong K_1$. So $|E(T)| = 0 = 1 - 1 = |V(T)| - 1$. Hence the result is true for all trees with 1 vertex

Ind. Step: Suppose the result is true for all trees with $\le p$ vertices. Let T be any tree with $p+1$ vertices. Choose any edge $e \in E(T)$. Then $T - \{e\}$ will be a disjoint union of two trees T_1 & T_2 because there was only one path between the endpoints of e. $\therefore |E(T)| = |E(T_1)| + |E(T_2)| + 1 = (|V(T_1)| - 1) + (|V(T_2)| - 1) + 1 = |V(T)| - 1$. So if the result is true for all trees with $\le p$ vertices, it will be true for all trees with $p+1$ vertices. By the Strong Princ. of Math Ind., the result now follows for all trees.

5(a) $d(u,v) = \begin{cases} \text{length of the shortest directed path from } u \text{ to } v \text{ in } G \\ \infty, \text{ if there is no directed path from } u \text{ to } v \text{ in } G. \end{cases}$

(b) Suppose $\deg(v) \geq (p-1)/2$ for each $v \in V(G)$. Now assume that G is a disconnected graph. Then we can find two vertices x_0 & y_0 such that there is no path from x_0 to y_0 in G. Let $A = \{v \in V(G) : v \text{ is adj. to } x_0\}$ & $B = \{v \in V(G) : v \text{ is adj. to } y_0\}$. Then $A \cap B = \emptyset$ because x_0 cannot be adjacent to y_0 and $A \cup B \subseteq V(G) - \{x_0, y_0\}$. So $|A \cup B| \leq p-2$. Since $A \cap B = \emptyset$, $\deg(x_0) + \deg(y_0) = |A| + |B| = |A \cup B| \leq p-2$. Hence $\deg(x_0) \leq (p-2)/2$ or $\deg(y_0) \leq (p-2)/2$ [because if $\deg(x_0) > (p-2)/2$ and $\deg(y_0) > (p-2)/2$, then $\deg(x_0) + \deg(y_0) > \frac{p-2}{2} + \frac{p-2}{2} = p-2$ which contradicts the fact that $\deg(x_0) + \deg(y_0) \leq p-2$]. But this contradicts the fact that $\deg(v) \geq (p-1)/2 > (p-2)/2$ for all $v \in V(G)$. $\therefore$ G must be a connected graph.

6 (a) A source-separating set of vertices $\mathcal{U}$ is any subset $\mathcal{U}$ of $V(G)$ such that $s \in \mathcal{U}$ and $t \notin \mathcal{U}$. $c[\mathrm{Cut}(\mathcal{U})] = \sum_{e \in \mathrm{Out}(\mathcal{U})} c(e)$, where $\mathrm{Out}(\mathcal{U}) = \{\vec{e} \in E(G) : e \text{ goes from } \mathcal{U} \text{ to } V(G) - \mathcal{U}\}$.

(b) Let G be a non trivial graph. Then $p(G) = |V(G)| \geq 2$. Now there are 2 cases: G has a vertex of degree 0, or G does not.

Case (i): G does not: In this case the only possible degrees in the G are $1, 2, 3, \ldots, p-1$ (because G has no vertices of degree 0 and the max. possible degree in G is $p-1$. Since we have p vertices and only $p-1$ possibilities, it follows by the Pigeon Hole Principle that G must have vertices of the same degree.

Case (ii) G has a vertex of deg. 0, say v_0.
In this case the only possible degrees in G are $0, 1, 2, \ldots, p-2$ because no vertex can be adj. to v_0. Hence by the PHP again, G has 2 vertices of the same degree. So in either case the result is true.