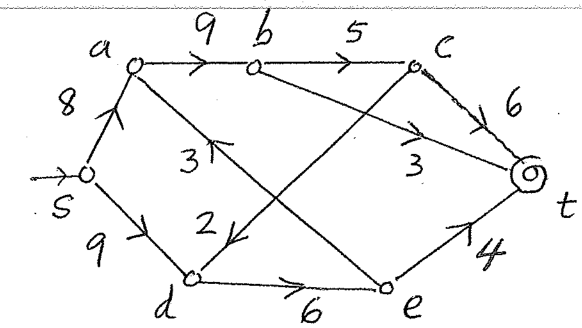
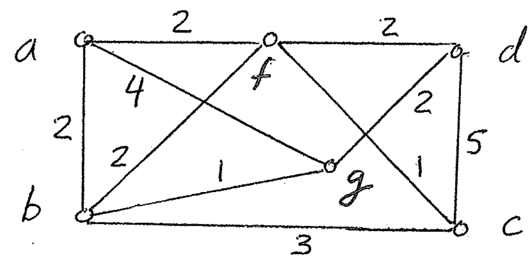
*Answer **all 6** questions.* ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

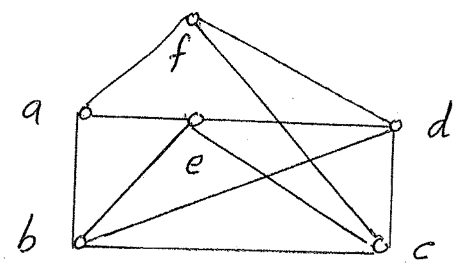
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

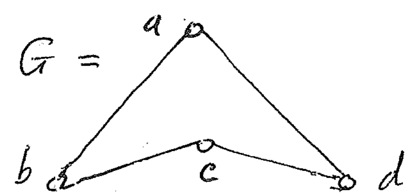


(18) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm*.

(b) If the graph H has no odd cycles, prove that $\chi(H) \leq 2$.



(15) 5(a) Let G be a graph. Define what is a *maximal path* of G.

(b) If $\delta(G) = 4$, prove that the graph G has a cycle of length at least 5.

(15) 6(a) Define what is a *convex polyhedron*.

(b) Let $\mathcal{E}$ be a planar embedding of a connected planar-graph G in which each region is bounded by at least 8 edges. Prove that $q \leq 4(p-2)/3$.

[*You may use any theorem that was proved in class for Qu. #6, if needed.*]

MAD 3305 - Graph Theory Florida International Univ.

Solutions to Test #2 Spring 2017

1. 1st aug. semi-path:

$s \xrightarrow{(0,8)} a \xrightarrow{(0,9)} b \xrightarrow{(0,5)} c \xrightarrow{(0,6)} t$

Slacks: 8 9 5 6 $\mu_1 = 5$

2nd aug. semi-path:

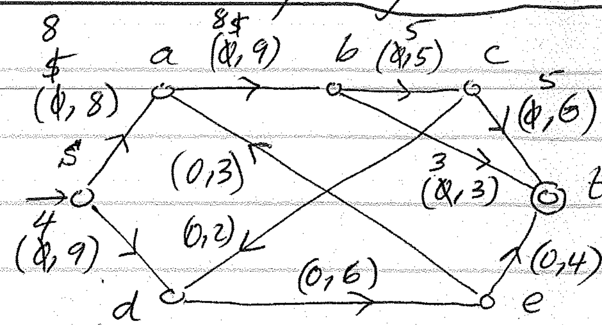
$s \xrightarrow{(5,8)} a \xrightarrow{(5,9)} b \xrightarrow{(0,3)} t$

slacks: 3 4 3 $\mu_2 = 3$

3rd aug. semi-path:

$s \xrightarrow{(0,9)} d \xrightarrow{(0,6)} e \xrightarrow{(0,4)} t$

Slacks 9 6 4 $\mu_3 = 4$



$S^* = \{u \in V(G)$: there is an aug. semi-path from s to $u\} = \{s, d, e, a, b\}$

$c(S^*) = c(\overrightarrow{bc}) + c(\overrightarrow{bt}) + c(\overrightarrow{et})$
$= 5 + 3 + 4 = 12$

$Val(f^*) = f^*(\overrightarrow{ct}) + f^*(\overrightarrow{bt}) + f^*(\overrightarrow{et}) = 5+3+4 = 12 = c(S^*)$ done!

2.

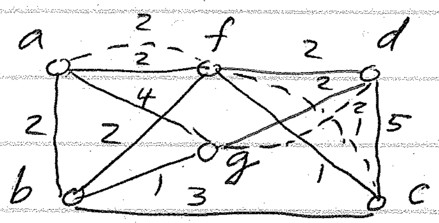
	a	c	d	g
a	.	3	4	3
c		.	3	4
d			.	2
f				.

choose

$\{a,c\} + \{d,g\} = 3+2 = 5$

$\{a,d\} + \{c,g\} = 4+4 = 8$

$\{a,g\} + \{c,d\} = 3+3 = 6$

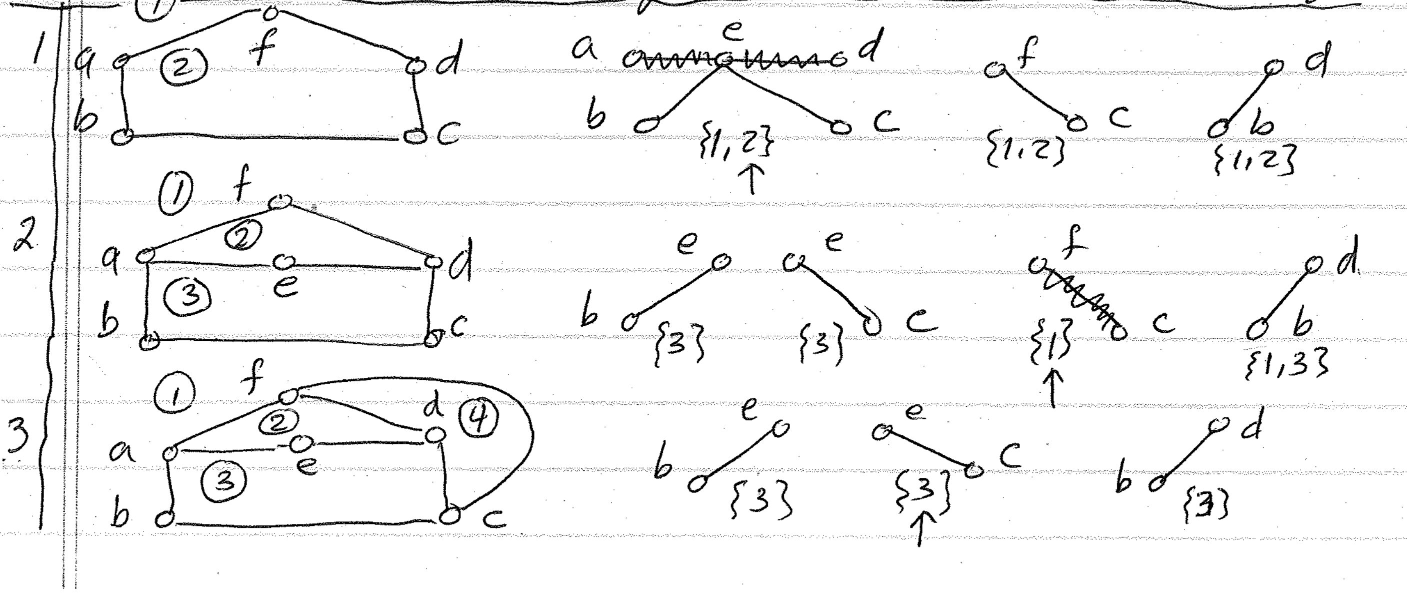


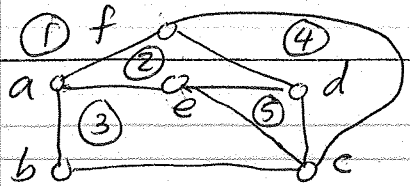
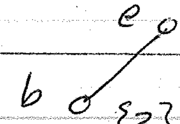
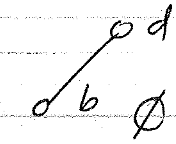
A minimum postman walk is

$a \overset{2}{—} f \overset{1}{—} c \overset{1}{—} f \overset{2}{—} a \overset{4}{—} g \overset{2}{—} d \overset{2}{—} g \overset{1}{—} b \overset{2}{—} f \overset{2}{—} d \overset{5}{—} c \overset{3}{—} b \overset{2}{—} a$

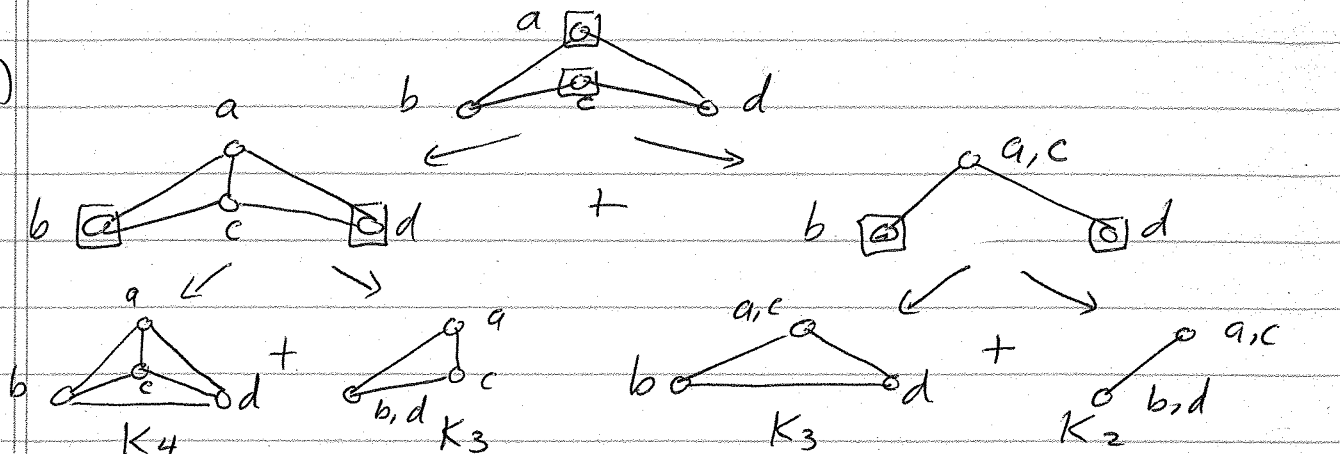
Length of our min. postman walk $= 2+1+1+2+4+2+2+1+2+2+5+3+2 = \boxed{29}$

3. i H_i Segments of G relative to H_i



3. i	H_i	Segments of G relative to H_i		
4		 $\{3\}$	 $\emptyset$	NON-PLANAR

4. (a)



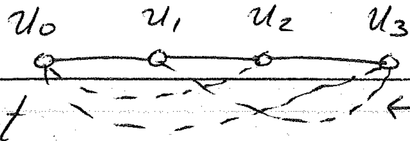
$$P_G(\lambda) = P_{K_4}(\lambda) + 2P_{K_3}(\lambda) + P_{K_2}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2)$$
$$+ \lambda(\lambda-1) = \lambda(\lambda-1)\left[(\lambda-2)(\lambda-3) + 2(\lambda-2) + 1\right] = \lambda(\lambda-1)(\lambda^2-3\lambda+3).$$

4(b) Let $F = T_1 \cup T_2 \cup \ldots \cup T_k$ be a spanning forest of H. Here $T_1, \ldots, T_k$ are the disjoint trees of F. Now select one vertex v_i from each T_i and designate it as the root of T_i. Then color the even levels of each T_i with color #1 & the odd levels of each T_i with color #2. Finally add back the edges of $E(H) - E(F)$, one at a time. Each time we add an edge, it must connect two vertices of different colors — otherwise we would get an odd cycle. Hence the coloring of F will be a legal coloring of H also. Thus $\chi(H) \leq 2$.

5 (a) A maximal path of G is any path P in G such that there is no other path P' of G which properly contains P.

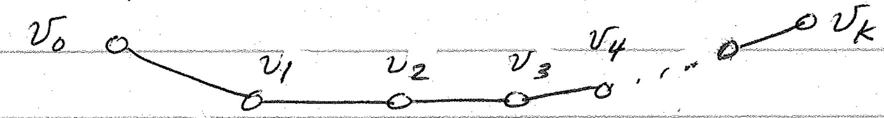
(b) We will show that any maximal path in G has length at least 4. Suppose there is a maximal path P_0 with with length at most 3. Then the endpoints of P_0 will have degree ≤ 3, because they cannot be adjacent to any

5(b) vertex outside P_0.



only possible edges out of u_3

But this contradicts the fact that $\delta(G) = 4$ (this means that each vertex has deg. ≥ 4). So any maximal path in G has length ≥ 4. Let $P = \langle v_0, \dots, v_k \rangle$ be a maximal path in G. Then $k \geq 4$.

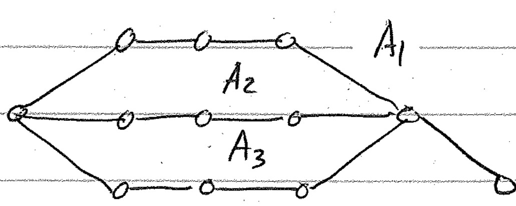


Now v_0 cannot be adjacent to any vertex outside $\{v_1, \dots, v_k\}$ otherwise P won't be maximal. So v_0 must be adjacent to v_i for some $i \geq 4$. (If v_0 was adj. only to vertices in $\{v_1, v_2, v_3\}$ then $\deg(v_0)$ would be ≤ 3.) But then $\langle v_0, v_1, v_2, \dots, v_{i-1}, v_i, v_0 \rangle$ would be a cycle of length $i+1$ which is ≥ 5. Hence G has a cycle of length ≥ 5.

6(a) A polyhedron Q is convex if any line that joins two points of Q always lies inside Q. (A polyhedron is a solid figure which is bounded by plane polygonal faces.)

(b) Let $A_1, A_2, \dots, A_r$ be the regions into which $\mathcal{E}$ partitions the plane. Then $8 \leq e(A_i)$ for each i. So $8r \leq e(A_1) + e(A_2) + \dots + e(A_r) \leq 2q$ because an edge can be counted in at most 2 regions. So $8r \leq 2q$.
$\therefore$ $4r \leq q$. Since G is connected $r = q + 2 - p$. Hence
$4(q+2-p) \leq q$. $\therefore$ $4q - q \leq 4(p-2)$
$\therefore$ $3q \leq 4(p-2)$. Hence $q \leq 4(p-2)/3$. END

Ex.



$r = 3$, $e(A_1) = 9$, $e(A_2) = 8$
$e(A_3) = 8$, $q = 13$, $p = 12$

$\underset{8 \cdot r}{8(3)} \leq \underbrace{9 + 8 + 8}_{25} \leq \underset{2 \cdot q}{2(13)}$

$13 \leq 4(12-2)/3$ ✓
$q \leq 4(p-2)/3$ ✓