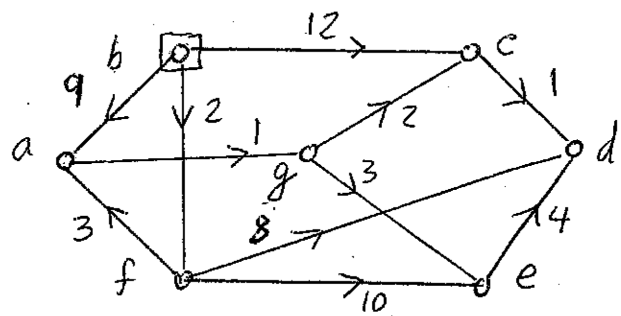
MAD 3305 - GRAPH THEORY
TEST #1 - FALL 2018

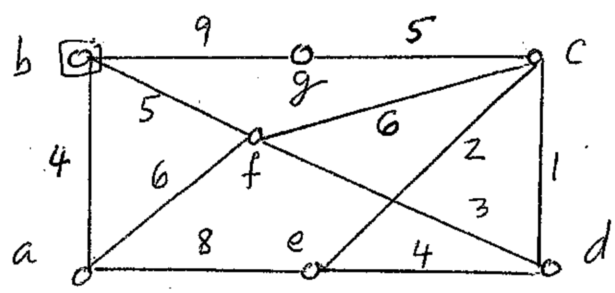
FLORIDA INT'L UNIV.
TIME: 75 min.

Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the *distances* from b to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2(a) Find a *graph* with degree sequence $\langle 4,3,3,2,2,2 \rangle$ by using the *Graphical Sequence Algorithm.*
(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* and starting at b.



(20) 3(a) Find the *tree* that corresponds to the sequence $\langle 1, 6, 6, 2 \rangle$ via *Prufer's Tree Decoding Algorithm.*
(b) The five characters *a, b, c, d, e* occur with frequencies *5, 6, 15, 4, 20;* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4 (a) Define what is the *distance* from u to v in a *weighted digraph G.*
(b) Prove that in any tree $T = \langle V(T),E(T) \rangle$ we always have $|E(T)| = |V(T)| - 1$.

(15) 5 (a) Define what is a *connected-component* H of a graph G
(b) Let *G* be a graph such that $deg(x) + deg(y) \geq p-1$ for any pair of non-adjacent vertices. Prove that G has only one connected-component.

(15) 6 (a) Define what is a *source-separating set U of vertices* in a network $N = \langle G,s,t,c \rangle$ and define what is the *capacity of the cut* determined by *U.*
(b) Prove that in any 4-ary tree with p vertices, $h(T) + 1 \geq (1/2).\log_2(3p+1)$.

MAD 3305 – Graph Theory Florida Int'l Univ.

Solutions to Test #1 Spring 2018

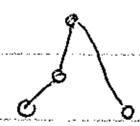
1.

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	v_0
∞	$\underline{0}$	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	b
9	.	12	∞	∞	$\underline{2}$	∞	$\{a,c,d,e,f,g\}$	1	f
$\underline{5}$	.	12	10	12	.	∞	$\{a,c,d,e,g\}$	2	a
.	.	12	10	12	.	$\underline{6}$	$\{c,d,e,g\}$	3	g
.	.	$\underline{8}$	10	9	.	.	$\{c,d,e\}$	4	c
.	.	.	$\underline{9}$	9	.	.	$\{d,e\}$	5	d
.	.	.	.	$\underline{9}$	.	.	$\{e\}$	6	e
.	.	.	.	.	.	.	$\emptyset$ STOP		

$d(b,\cdot) = 5 \quad 0 \quad 8 \quad 9 \quad 9 \quad 2 \quad 6.$

2 (a) $4, \underline{3, 3, 2, 2}$

$\quad 2, 2, 1, 1 \longrightarrow$



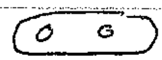
G_3

$\quad 1, 0, 1$

$\quad 1, \underline{1}, 0 \longrightarrow$

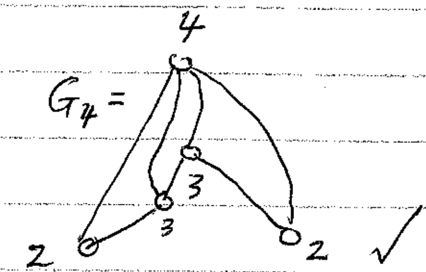
G_2

$\quad 0, 0 \longrightarrow$



G_1

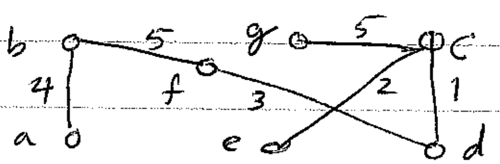
$G_4 =$



(b)

$E(T)$	$V(T)$	a	b	c	d	e	f	g	i	x_0
$\emptyset$	$\{b\}$	∞	$\underline{0}$	∞	∞	∞	∞	∞	0	b
$\{\overline{ba}\}$	$\{b,a\}$	$\underline{4}$	.	∞	∞	∞	5	9	1	a
$\{\overline{ba}, \overline{bf}\}$	$\{b,a,f\}$	.	.	∞	∞	8	$\underline{5}$	9	2	f
$\{\overline{ba}, \overline{bf}, \overline{fd}\}$	$\{b,a,f,d\}$	.	.	6	$\underline{3}$	8	.	9	3	d
$\{\overline{ba}, \overline{bf}, \overline{fd}, \overline{dc}\}$	$\{b,a,f,d,c\}$	.	.	$\underline{1}$	.	4	.	9	4	c
$\{\overline{ba}, \overline{bf}, \overline{fd}, \overline{dc}, \overline{ce}\}$	$\{b,a,f,d,c,e\}$	.	.	.	.	$\underline{2}$	.	5	5	e
$\{\overline{ba}, \overline{bf}, \overline{fd}, \overline{dc}, \overline{ce}, \overline{cg}\}$	$\{b,a,f,d,c,e,g\}$	.	.	.	.	.	.	5	6	g

$T =$



$\omega(T) = 4+5+3+1+2+5 = 20$

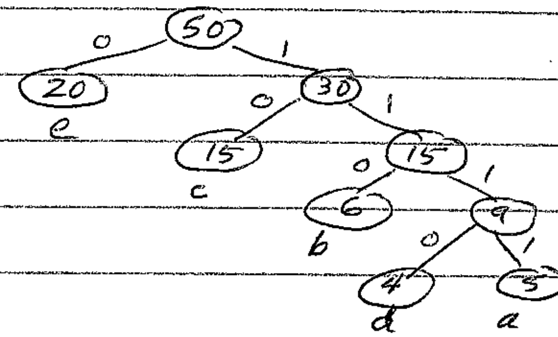
but this was not requested.

3(a) $\underset{\sim}{s} = \langle 1,6,6,2 \rangle$, so $p = |\underset{\sim}{s}| + 2 = 4 + 2 = 6$.

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$\ell(i)$ — $s(i)$
2	2	1	1	1	3	1	3 — 1
1	2	0	1	1	3	2	1 — 6
0	2	0	1	1	2	3	4 — 6
0	2	0	0	1	1	4	5 — 2
0	1	0	0	0	1	5	2 — 6 STOP

$\therefore$ $T =$ 

(b)
d a b c e
4, 5, 6, 15, 20
6, → 9, 15, 20
15, → 15, 20
20, → 30
50



Char	a	b	c	d	e
Freq.	5	6	15	4	20
Code	1111	110	10	1110	0
length	4	3	2	4	1

WPL (coding) $= 5(4) + 6(3) + 15(2) + 4(4) + 20(1)$
$= 20 + 18 + 30 + 16 + 20 = 104$.

4(a) $dist(u,v) = \begin{cases} \text{length of the shortest directed path from } u \text{ to } v \text{ in } G. \\ \infty \quad \text{if there is no directed path from } u \text{ to } v \text{ in } G \end{cases}$

(b) We shall prove the result by strong induction on $p = |V(G)|$.
<u>Basis</u> If $p=1$, then $T \cong K_1$. So $|E(T)| = 0 = 1-1 = |V(T)|-1$. Hence the result is true for $p=1$. <u>Ind. step</u>: Suppose the $|E(T)| = |V(T)|-1$ for all trees with $\leq p$ vertices. Let T_0 be any tree with $p+1$ vertices. Choose any edge $e \in E(T_0)$. Then $T_0 - \{e\}$ will consist of two disjoint trees T_1 & T_2. Hence
$|E(T_0)| = |E(T_1)| + |E(T_2)| + |\{e\}| = |V(T_1)-1| + |V(T_2)| - 1 + 1 = |V(T_0)| - 1$.
So if the result is true for all trees T with $|V(T)| \leq p$, it will be true for all trees T_0 with $|V(T_0)| = p+1$. <u>Concl.</u> So by the strong princ. of ind. it is true for all trees.

5(a) A connected-component of G is any connected subgraph H of G such that there is no connected subgraph H' of G which properly contains H.

(b) Suppose G is a graph such that for any pair of non-adjacent vertices X & Y, $deg(x) + deg(y) \geq p-1$. Here $p = |V(G)|$. No let us assume that G is disconnected. Then we can find two vertices x_0 & y_0 in different components of G. So there will be no path from x_0 to y_0. Let $A = \{v \in V(G): v \text{ is adj. to } x_0\}$ & $B = \{v \in V(G) : v \text{ is adj. to } y_0\}$ Then $A \cap B = \emptyset$ bec. there is no path from x_0 to y_0. Hence $deg(x_0) + deg(y_0) = |A| + |B| = |A \cup B| \leq |V(G) - \{x_0, y_0\}| \leq p-2$. Since x_0 & y_0 are non-adj., we should have $deg(x_0) + deg(y_0) \geq p-1$ which contradicts the fact that $deg(x_0) + deg(y_0) \leq p-2$. So G cannot a disconnected graph. Hence G must be a connected graph & consequently must have one component.

6(a). A source-separating set of vertices in N is any subset U of $V(G)$ with $s \in U$ & $t \notin U$. The capacity of the cut determined by U is defined by $c[Cut(U)] = \sum_{e \in Cut(U)} c(e)$, where $Out(U) = \{e \in E(G) : e \text{ goes from } U \text{ to } \bar{U}\}$

(b) Let T be any 4-ary tree and $k = h(T)$. Then T can have at most 4^0 vertices at level 0, at most 4^1 vertices at level 1, ... and at most 4^k vertices at level k. So

$$p = |V(G)| \leq 4^0 + 4^1 + 4^2 + \dots + 4^k = (4^{k+1} - 1)/(4-1)$$

$$\therefore \quad 3p \leq 4^{k+1} - 1. \quad \therefore \quad 3p+1 \leq 4^{k+1} = 2^{2(k+1)}.$$

$$\therefore \quad \log_2(3p+1) \leq \log_2 2^{2(k+1)} = 2(k+1)$$

$$\therefore \quad \tfrac{1}{2}\log_2(3p+1) \leq k+1. \quad \text{Hence } h(T)+1 \geq \tfrac{1}{2}\log_2(3p+1)$$

END.