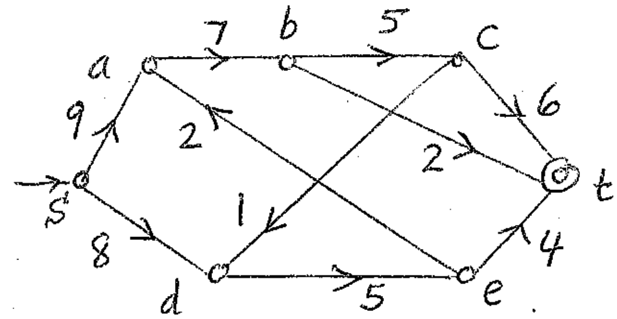
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.

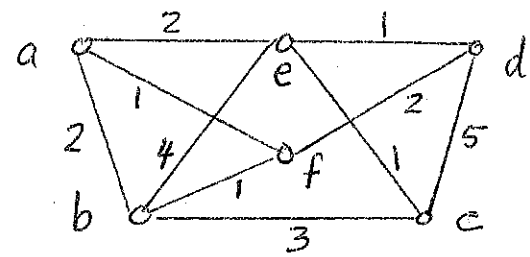
TEST #2 - SPRING 2018 TIME: 75 min.

*Answer **all 6** questions.* ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

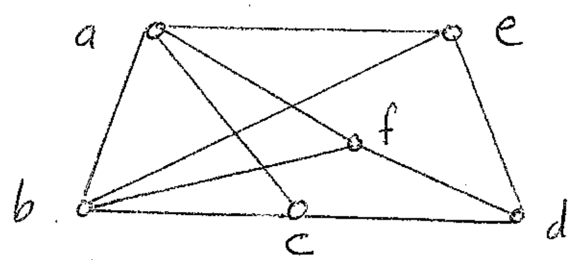
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

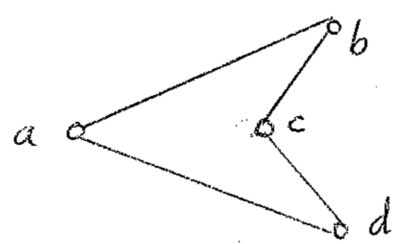


(18) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm.*

(b) If the graph H has no odd cycles, prove that $\chi(H) \leq 2$.



(15) 5(a) Let G be a connected multi-graph. Define what is an *open Euler-trail* of G.

(b) Prove G has an open Euler-trail if & only if G has exactly 2 vertices of odd degree. [*You may use the Euler-Circuit theorem in question #5, if needed.*]

(15) 6(a) Define what is a *polyhedral graph* G & what is its *dual graph* G^*.

(b) Let $\mathcal{E}$ be a planar embedding of a connected planar-graph G in which each region is bounded by at least 5 edges. Prove that $q \leq 5(p-2)/3$.

[*You may use any theorem that was proved in class for Qu. #6, if needed*

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Solutions to Test #2 — Spring 2018

#1. <u>First</u> aug. semi-path

$s \xrightarrow{(0,9)} a \xrightarrow{(0,7)} b \xrightarrow{(0,5)} c \xrightarrow{(0,5)} t$

<u>slacks</u> 9 7 5 5 $\boxed{M_1 = 5}$

<u>Second</u> aug. semi-path

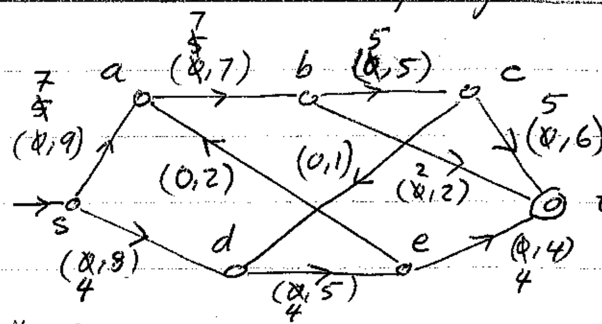
$s \xrightarrow{(5,9)} a \xrightarrow{(5,7)} b \xrightarrow{(0,2)} t$

<u>slacks</u> 4 2 2 $\boxed{M_2 = 2}$

<u>Third</u> aug. semi-path

$s \xrightarrow{(0,8)} d \xrightarrow{(0,5)} e \xrightarrow{(0,4)} t$

<u>Slacks</u> 8 5 4 $\boxed{M_3 = 4}$



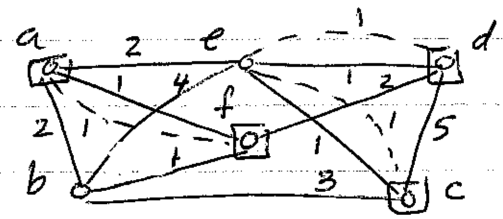
$S^* = \{u \in V:$ there is an aug. semi-path from s to u$\}$
$= \{s, a, d, e\}$

$c(S^*) = c(\vec{ab}) + c(\vec{et}) = 7 + 4 = 11$ ✓

$Val(f^*) = f^*(\vec{ct}) + f^*(\vec{bt}) + f^*(\vec{et}) = 5+2+4 = 11.$

#2

dist	a	c	d	f
a	.	3	3	1
c		.	2	4
d			.	2
f				.



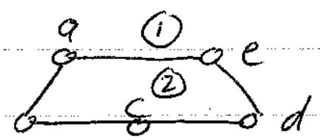
$\{a,c\} + \{d,f\}$ $3+2=5$ $\{a,d\}+\{c,f\}$ $3+4=7$ $\{a,f\}+\{c,d\}$ $1+2=3$ ✓

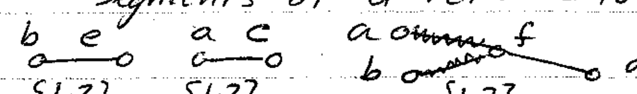
Min. postman walk

$= a \overset{2}{—} e \overset{1}{—} d \overset{1}{—} e \overset{1}{—} c \overset{1}{—} e \overset{4}{—} b \overset{3}{—} c \overset{5}{—} d \overset{2}{—} f \overset{1}{—} a \overset{1}{—} f \overset{1}{—} b \overset{2}{—} a$, length = 25.

#3. i H_i Segments of G relative to H_i

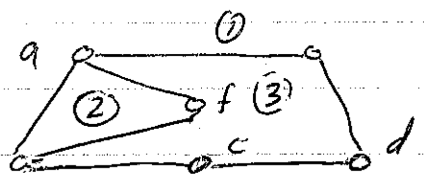
1

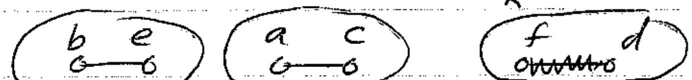




$R_i(s) =$ {1,2} {1,2} {1,2} ↑R

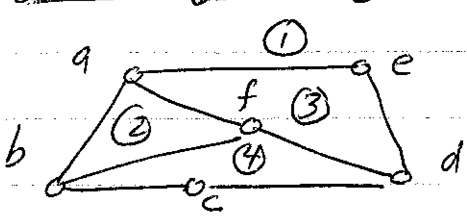
2





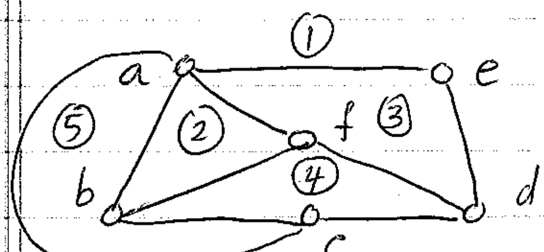
$R_i(s) = \{1,3\}$ {1,3} {3} ↑R

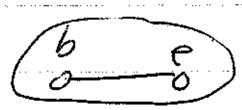
3.



$R_i(s) = \{1\}$ {1} ↑R

4.

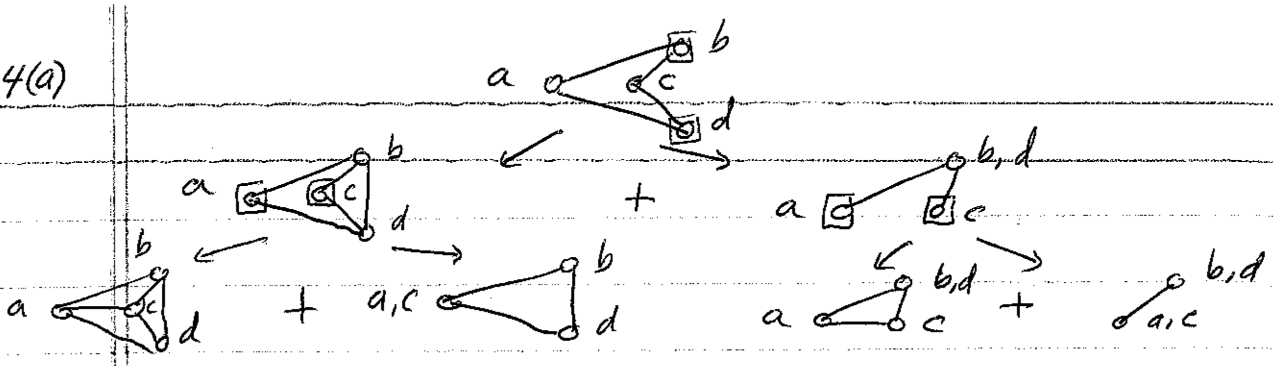




$R_i(s) = \emptyset$ NON-PLANAR

∴ G is non-planar.

4(a)



$\therefore\ P_G(\lambda) = P_{K_4}(\lambda) + 2P_{K_3}(\lambda) + P_{K_2}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)$

$= \lambda(\lambda-1)[(\lambda-2)(\lambda-3) + 2(\lambda-2) + 1] = \lambda(\lambda-1)[\lambda^2 - 3\lambda + 3]$.

(b) Let $\mathcal{F} = T_1 \cup T_2 \cup T_3 \cup \dots \cup T_k$ be a <u>spanning forest</u> of the k connected components of H. Choose a vertex v_i $(i = 1, \dots, k)$ in each T_i and let it be the root of T_i. Then each $\langle T_i, v_i \rangle$ be a rooted tree. Color the even levels with color #1 & the odd levels with color #2. Now put back the other edges of H, one at a time. Each time we add an edge we must join two vertices with different colors – otherwise, we would get an odd cycle in H. So H can be legally colored with 2 colors. $\therefore\ \chi(H) \le 2$.

5(a) An open Euler trail of G is any walk in G which includes each edge of G exactly once and with starting vertex $\neq$ terminal vertex.

(b) ($\Rightarrow$): Suppose G has an open Euler trail $\langle v_0, e_1, v_1, e_2, \dots, e_n, v_n \rangle$. Then $v_0 \neq v_n$. Now if we add a new edge e_{n+1} between v_0 & v_n we will get a new multi-graph G' with an Euler-circuit $\langle v_0, e_1, v_1, e_2, \dots, e_n, v_n, e_{n+1}, v_0 \rangle$. So by the Euler-Circuit Theorem each vertex of G' must be of even degree. But $G = G'$ – (the new edge we added), so v_0 & v_n will be the only vertices of odd degree in G.

($\Leftarrow$): Suppose G has exactly 2 vertices of odd degree. Add a new edge, e, between those two vertices of odd degree to get a new multi-graph G' with all vertices being of even degree. Since G' is connected, it follows from the Euler–circuit Theorem that G' has an Euler-circuit. Now if we remove the edge, e (that we added) – from this Euler circuit, we will get an open Euler-trail of G.

6(a) A polyhedral graph G is any graph that can be obtained by considering the vertices & edges of a convex polyhedron as vertices & edges of a graph. The dual G^* of G is defined as follows: Let V^* = set of polygonal faces of the polyhedron from which G came. Each time we have an edge of the polyhedron in common between two faces R & S, we get an edge $\underbrace{e_{RS} \text{ in } G^*}_{\text{between } R \& S}$. Let E^* = set of all such edges. Then $G^* = \langle V^*, E^* \rangle$.

(b) Let $A_1, A_2, \ldots, A_r$ be the regions into which $\mathcal{E}$ partitions the plane. Then the number of edges, $e(A_i)$ in each A_i will satisfy $e(A_i) \geq 5$. So if $q = |E(G)|$ & $p = |V(G)|$,

$$5r \leq e(A_1) + e(A_2) + \ldots + e(A_r) \leq 2q$$

because each edge is counted at most twice.

So $5r \leq 2q$. But G is a connected planar graph, so $r = q + 2 - p$. $\therefore$ $5(q+2-p) \leq 2q$. So $5q + 10 - 5p \leq 2q$

$\therefore$ $3q \leq -10 + 5p$ $\therefore$ $q \leq 5(p-2)/3$ END