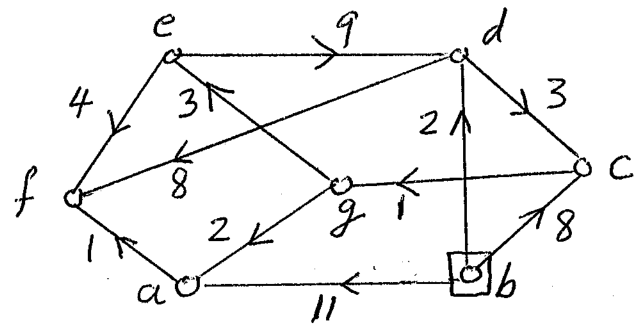
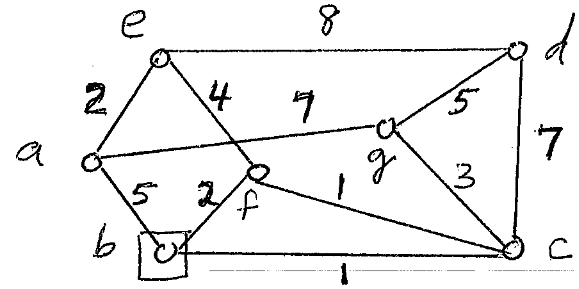
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.
TEST #1 - SPRING 2020 TIME: 75 min.

Answer *all 6* questions. ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON A SEPARATE PAGE.*

(15) 1. Find the *distances* from b to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2 (a) Find a *graph* with degree sequence $\langle 5,3,2,2,2,2 \rangle$ by using the *Graphical Sequence Algorithm.*
(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* and starting at b.



(20) 3 (a) Find the *tree* that corresponds to the sequence $\langle 4, 1, 2, 4 \rangle$ via the *Prufer's Tree Decoding Algorithm.*
(b) The five characters *a, b, c, d, e* occur with frequencies *20, 3, 5, 14, 8;* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4 (a) Define what is the *adjacency matrix* of a *digraph G* with $V(G) = \{1,2,3,\ldots,p\}$.
(b) Prove that $A^n[i,j]$ = number of *directed walks* of length n from i to j in G.

(15) 5 (a) Define what is the *distance,* $d(u,v)$, from u to v in a *weighted digraph G.*
(b) If G is a graph with p vertices and $|E(G)| > (p-1).(p-2)/2$, prove that G will always be a connected graph.

(15) 6 (a) Define what it means for the *digraph* G to be *isomorphic* to the *digraph* H.
(b) A certain tree *T* has 8 vertices of degree 5, 10 of degree 4, 6 of degree 3, and the rest of deg. 1 or 2. What is the smallest number of *vertices* that *T* can have?
[*You may use any theorem that was proved in class to answer question #6.*]

MAD 3305 – Graph Theory — Florida International Univ.

Solutions to Test # 1 — SPRING 2020

1.

	$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	$T(i)$	i	$V_0(i)$
	∞	0	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	$b \to a,c,d$
	11	.	8	2	∞	∞	∞	$\{a,c,d,e,f,g\}$	1	$d \to c,f$
	11	.	5	.	∞	10	∞	$\{a,d,e,f,g\}$	2	$c \to g$
	11	.	.	.	∞	10	6	$\{a,e,f,g\}$	3	$g \to a,e$
	8	.	.	.	9	10	.	$\{a,e,f\}$	4	$a \to e$
	.	.	.	.	9	9	.	$\{e,f\}$	5	$e \to f$
	.	.	.	.	.	9	.	$\{f\}$	6	f
$d(b,\cdot) =$	8	0	5	2	9	9	6	$\emptyset$ STOP		

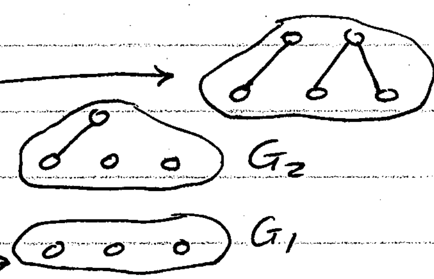
2(a) 5, 3, 2, 2, 2, 2

2, 1, 1, 1, 1 $\longrightarrow$ G_3

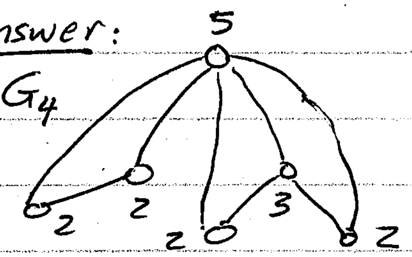
0, 0, 1, 1 $\longrightarrow$ G_2

1, 1, 0, 0

0, 0, 0 $\longrightarrow$ G_1



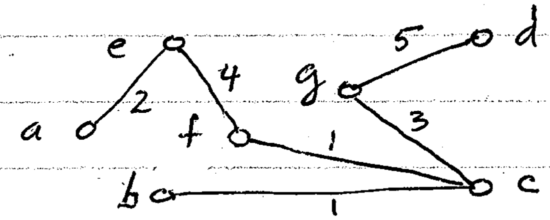
Answer: G_4



(b)

$E(T)$	$V(T)$	a	b	c	d	e	f	g	i	$X_0(i)$
$\emptyset$	$\{b\}$	∞	0	∞	∞	∞	∞	∞	0	b
$\{\overline{bc}\}$	$\{b,c\}$	5	.	1	∞	∞	2	∞	1	c
$\{\overline{bc},\overline{cf}\}$	$\{b,c,f\}$	5	.	.	7	∞	1	3	2	f
$\{\overline{bc},\overline{cf},\overline{cg}\}$	$\{b,c,f,g\}$	5	.	.	7	4	.	3	3	g
$\{\overline{bc},\overline{cf},\overline{cg},\overline{fe}\}$	$\{b,c,f,g,e\}$	5	.	.	5	4	.	.	4	e
$\{\overline{bc},\overline{cf},\overline{cg},\overline{fe},\overline{ea}\}$	$\{b,c,f,g,e,a\}$	2	.	.	5	.	.	.	5	a
$\{\overline{bc},\overline{cf},\overline{cg},\overline{fe},\overline{ea},\overline{gd}\}$	$\{b,c,f,g,e,a,d\}$	.	.	.	5	.	.	.	6	d

Answer: $T =$



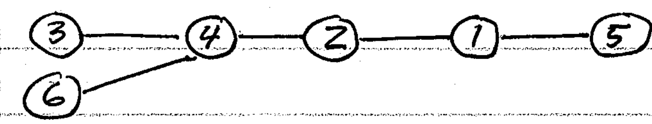
$\omega(T) = 1+1+2+3+4+5$

$= 16$

3(a) $\underset{\sim}{s} = \langle 4,1,2,4 \rangle$, so $p = |\underset{\sim}{s}| + 2 = 4 + 2 = 6$.

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$\ell(i)$	$s(i)$
2	2	1	3	1	1	1	3 —	4
2	2	0	2	1	1	2	5 —	1
1	2	0	2	0	1	3	1 —	2
0	1	0	2	0	1	4	2 —	4
0	0	0	1	0	1		4 —	6

$T =$



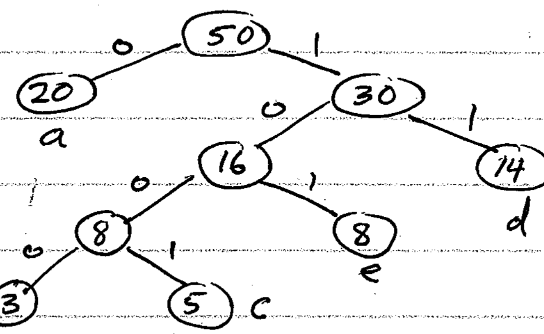
(b) 3, 5, 8, 14, 20

8, 8, 14, 20

14, 16, 20

20, 30

50



Char	a	b	c	d	e
Freq.	20	3	5	14	8
Code	0	1000	1001	11	101
length	1	4	4	2	3

W.P.L. of the coding

$= 20(1) + 3(4) + 5(4) + 14(2) + 8(3)$

$= 20 + 12 + 20 + 28 + 24 = \boxed{104}$

4(a) The adjacency matrix of the digraph G is defined by
$A[i,j]$ = No. of directed edges from vertex i to vertex j.

(b) We will prove the result by induction on n.

Basis: If $n=1$, then no. of directed walks of length 1 from i to j = no. of directed edges from i to j $= A^1[i,j]$. So result is true for $n=1$.

Ind. Step: Suppose the result is true for n (with any i & j). Then no. of directed walks of length n from any i to any k $= (A^n)[i,k]$. So

$$\begin{pmatrix}\text{No. of directed walks of} \\ \text{length } n+1 \text{ from } i \text{ to } j \text{ in } G\end{pmatrix} = \sum_{k=1}^{p} \begin{pmatrix}\text{No. of dir. walks of} \\ \text{length } n \text{ from } i \text{ to } k\end{pmatrix} \cdot \begin{pmatrix}\text{no. of dir. walks of} \\ \text{length 1 from } k \text{ to } j\end{pmatrix}$$

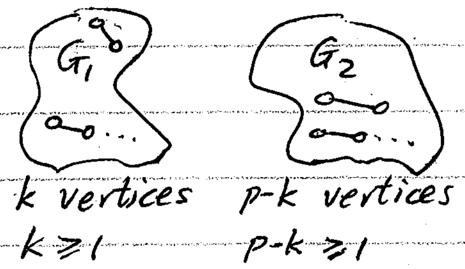
$$= \sum_{k=1}^{p} (A^n)[i,k] \cdot (A^1)[k,j] = (A^{n+1})[i,j].$$

So if the result is true for n, it will be true for $n+1$.

Concl: By the Principle of Math Induction, the result follows for all $n \in \mathbb{N}$

5(a) $d(u,v) = \begin{cases} \text{length of the shortest directed path from } u \text{ to } v \text{ in } G \\ +\infty \quad \text{if there is no directed path from } u \text{ to } v \text{ in } G \end{cases}$

(b) Suppose G is not a connected graph. Then we can split G into two parts G_1 & G_2 such that there are no edges between G_1 & G_2. Let $|V(G_1)| = k$. Then $|V(G_2)| = p-k$. So



$$|E(G)| = |E(G_1)| + |E(G_2)|$$
$$\leq k(k-1)/2 + (p-k)(p-k-1)/2.$$

$$\therefore \frac{(p-1)(p-2)}{2} - |E(G)| \geq \{(p-1)(p-2) - k(k-1) - (p-k)(p-k-1)\}/2$$
$$= \{p^2 - 3p + 2 - k^2 + k - p^2 + pk + pk - k^2 + p - k\}/2$$
$$= (2pk - 2k^2 - 2p + 2)/2 = pk - k^2 - p + 1$$
$$= kp - k^2 - k - p + k + 1 = (k-1)(p-k-1) \geq 0$$

$\therefore |E(G)| \leq (p-1)(p-2)/2$ — which contradicts the fact that we were given that $|E(G)| > (p-1)(p-2)/2$. $\therefore$ G must be connected.

6(a) G is isomorphic to H if we can find a bijection $\alpha: V(G) \to V(H)$ such that $\langle u, v \rangle \in E(G) \Leftrightarrow \langle \alpha(u), \alpha(v) \rangle \in E(H)$.

(b) Let $p = |V(T)|$, ℓ = no. of leaves in T, and k = no. of vertices of degree 2 in T. Now we know from the class that $|E(T)| = |V(T)| - 1 = p - 1$, and sum of degrees in $T = 2|E|$

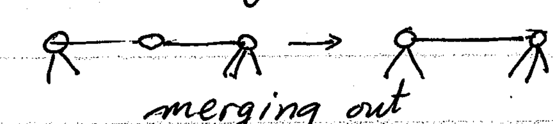
$\therefore$ sum of degrees in $T = 2(p-1) = 2p - 2$.

Also $p = 8 + 10 + 6 + k + \ell = 24 + k + \ell$

$$\therefore \quad 8(5) + 10(4) + 6(3) + k(2) + \ell(1) = 2(\overbrace{24 + k + \ell}^{p}) - 2$$

$$\therefore \quad 98 + 2k + \ell = 48 + 2k + 2\ell - 2 \qquad \therefore \ \ell = 52$$

Now if we "merge out" all the vertices of degree 2, we will get a tree T' with zero vertices of degree 2 and in which all the other degrees are the same. So the smallest no. of vertices that T can have will be $24 + \ell = 24 + 52 = \boxed{76}$



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