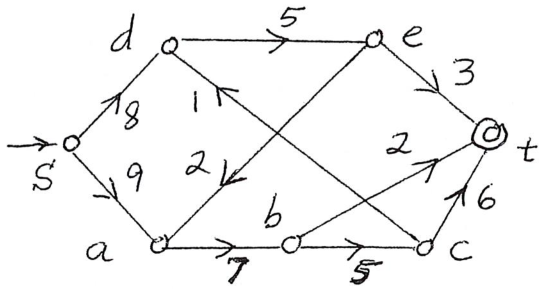
MAD 3305 - GRAPH THEORY | FLORIDA INT'L UNIV.

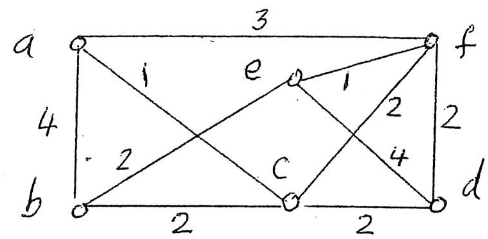
TEST #2 - SPRING 2020 | TIME: 75 min.

*Answer **all 6** questions.* ***No Calculators or Cellphones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

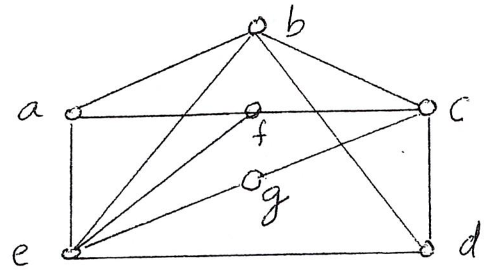
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm.* Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

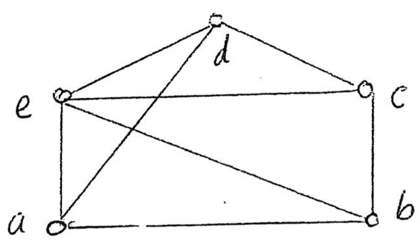


(18) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm.* [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find $P_G(\lambda)$ for the graph *G* on the right by using the *Chromatic Polynomial Algorithm.*

(b) Prove in any graph G, $\chi(G) \leq \Delta(G)+1$.



(15) 5 (a) Explain in your own words what is the difference between of a *minimum postman walk* and a *minimum salesman walk* in a weighted multi-graph G.

(b) Write down Ore's theorem & use it to prove that any graph G with $\deg(x) + \deg(y) \geq p-1$ for all pairs of non-adjacent vertices x &y, has a *Hamilton path.*

(15) (a) Explain what is the difference between a *polyhedral graph* G & its *dual* G^*. Is it possible for G and G^* to be *isomorphic* – explain your answer.

(b) Let $\mathcal{E}$ be a planar embedding of a connected planar-graph G in which each region is bounded by at least 10 edges. Prove that $4q \leq 5(p-2)$.

[*You may use any theorem that was proved in class for Qu. #6, if needed*]

MAD 3305 - Graph Theory Florida International Univ

Solutions to Test #2 Spring 2020

#1. 1st aug. semi-path

$s \xrightarrow{(0,8)} d \xrightarrow{(0,5)} e \xrightarrow{(0,3)} t$

Slacks 8 5 3 $\mu_1 = 3$

2nd aug. semi-path

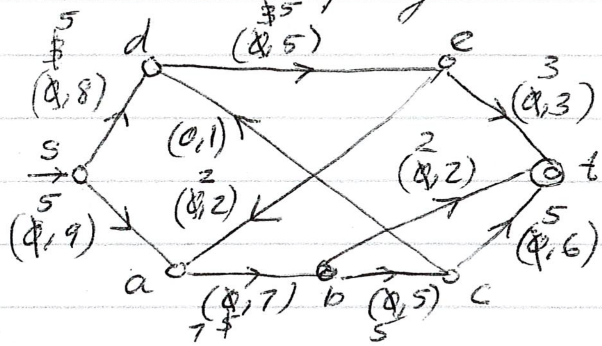
$s \xrightarrow{(0,9)} a \xrightarrow{(0,7)} b \xrightarrow{(0,5)} c \xrightarrow{(0,6)} t$

Slacks 9 7 5 6 $\mu_2 = 5$

3rd aug. semi-path

$s \xrightarrow{(3,8)} d \xrightarrow{(3,5)} e \xrightarrow{(0,2)} a \xrightarrow{(5,7)} b \xrightarrow{(0,2)} t$

Slacks 5 2 2 2 2 $\mu_3 = 2$



$S^* = \{s, d, a, e\}$

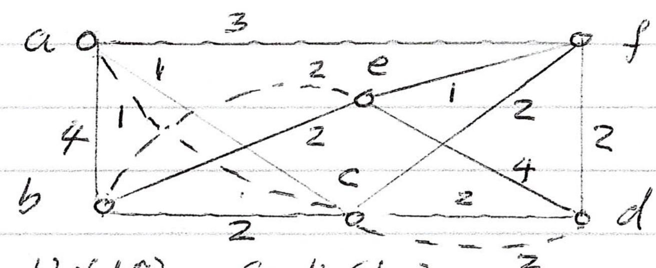
$c(S^*) = c(\vec{ab}) + c(\vec{et})$

$= 7 + 3 = 10$

$Val(f^*) = \text{netflow into } t = 3+2+5 = 10 = c(S^*)$ ✓

#2.

dist.	a	b	d	e
a	.	3	3	4
b	.	.	4	2
d	.	.	.	3
e	.	.	.	.



$\{a,b\}+\{d,e\}$ $3+3=6$

$\{a,d\}+\{b,e\}$ $3+2=5$ ✓

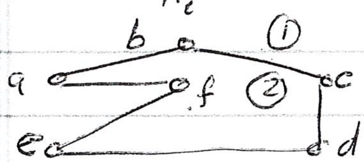
$\{a,e\}+\{b,d\}$ $4+4=8$

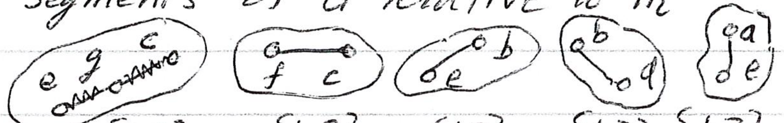
A minimum postman walk is

$a \xrightarrow{1} c \xrightarrow{2} d \xrightarrow{2} c \xrightarrow{1} a \xrightarrow{4} b \xrightarrow{2} e \xrightarrow{2} b \xrightarrow{2} c \xrightarrow{2} f \xrightarrow{2} d \xrightarrow{4} e \xrightarrow{1} f \xrightarrow{3} a$

length = 28

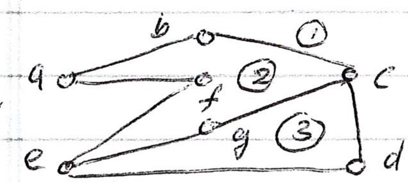
#3. i H_i segments of G relative to H_i

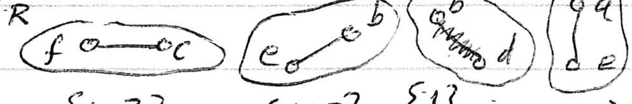
1 



$R_1 = \{1,2\}, \{1,2\}, \{1,2\}, \{1,2\}, \{1,2\}$

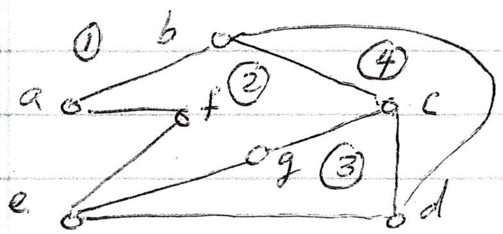
↑ R

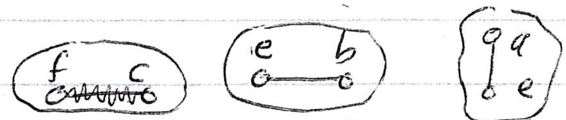
2. 



$R_2 = \{1,2\}$ $\{1,2\}$ $\{1\}$ $\{1,2\}$

↑ R

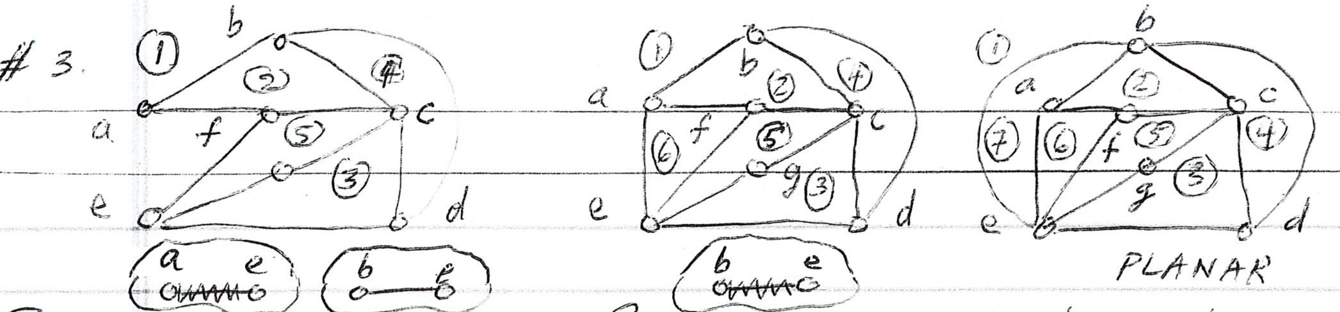
3. 



$R_3 = \{2\}$ $\{1,2\}$ $\{1,2\}$

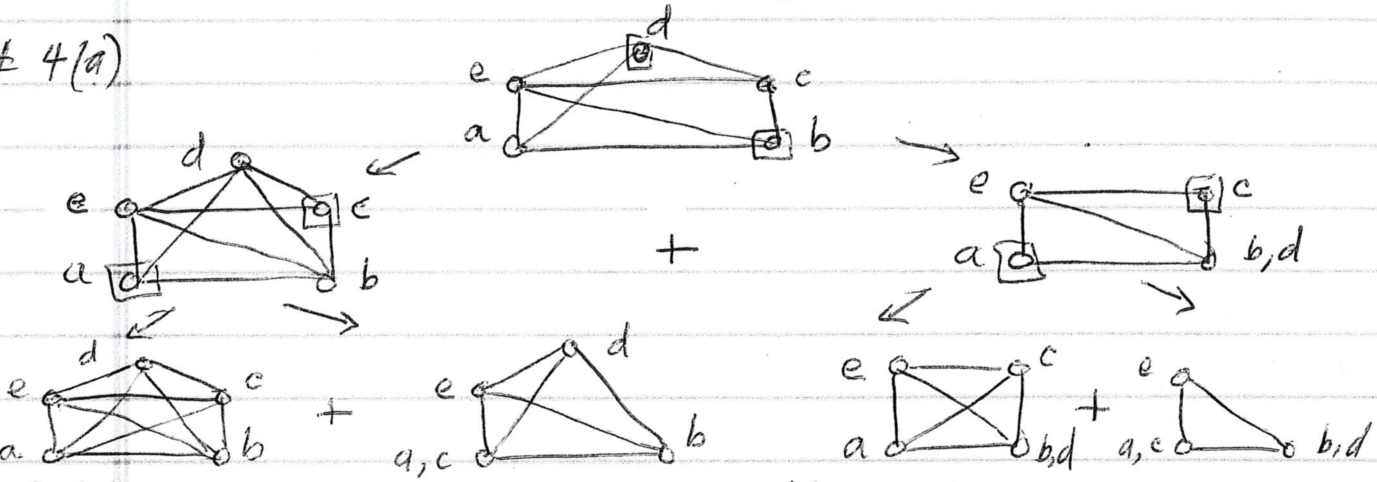
↑ R

\# 3.



$R_4 = \{1\}, \{1\}$ ↑R $\quad R_5 = \{1\}$ ↑R $\quad$ Graph is planar.

\# 4(a)



$$P_G(\lambda) = P_{K_5}(\lambda) + 2P_{K_4}(\lambda) + P_{K_3}(\lambda)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1] = \lambda(\lambda-1)(\lambda-2)[\lambda^2-5\lambda+7].$$

4(b) We will prove the result by induction on $p = |V(G)|$.

Basis: If $p=1$, then $G \cong K_1$ and so $\chi(G) = 1 \le 0+1 = \Delta(G)+1$. Result is true for $p=1$

Ind. Step: Suppose the result is true for all graphs with p vertices. Let G be any graph with $p+1$ vertices. Choose any vertex, v_0, in $V(G)$ and let $H = G - \{v_0\}$. Then $\Delta(H) \le \Delta(G)$ and since H has p vertices, H can be legally colored with $\Delta(G)+1$ colors. Now we know that v_0 is adjacent to at most $\Delta(G)$ vertices in G (because $\Delta(G)$ = largest degree in G), and since there are $\Delta(G)+1$ colors available, we can always find a color for v_0 which is different from that of all its adj. vertices. Hence $\chi(G) \le \Delta(G)+1$. So if the result is true for p, it will be true for $p+1$.

Conclusion: So by the Principle of Mathematical Induction the result is true for all graphs.

#5(a) A minimum postman walk is a closed walk of G that contains every edge of G & is of minimum total length while a minimum salesman walk is a closed walk of G that contains every vertex of G and is of minimum total length.

5(b) Ore's Theorem: If G is a graph with $p \geq 3$ vertices & for any pair of non-adj. vertices x & y, $deg(x) + deg(y) \geq p$, then G has a Hamilton cycle.
Let G be a graph with p vertices such that for any pair of non-adj. vertices x & y, $deg(x) + deg(y) \geq p-1$. Now let H be the graph obtained by add a new vertex v_{p+1} and edges from v_{p+1} to all the vertices of G. Then for any pair of non-adj. vertices x & y in H, $deg_H(x) + deg_H(y) = deg_G(x)+1 + deg_G(y)+1 \geq (p-1)+2 = p+1$.
So by Ore's theorem, H has a Hamilton cycle, C. Now if we remove the vertex v_{p+1} from C, we'll get a Hamilton path in G.

#6(a) A polyhedral graph G is a graph that is obtained by considering the vertices & edges of a polyhedron P as vertices & edges of a graph — while the dual G^* of G is the graph obtained by considering the faces of P as vertices and for each edge that is shared by two faces, there will be an edge in G^*. Yes, it is possible to have $G \cong G^*$ — just take G to be K_4 = tetrahedral graph

6(b) Let $A_1, A_2, \ldots, A_r$ be regions into which the plane is partitioned by E (the embedding of G). Then

$$10r \leq e(A_1) + e(A_2) + \cdots + e(A_r) \leq 2q$$

because $e(A_i) \geq 10$ for each i & each edge can be counted in at most 2 regions. So $10r \leq 2q$ $\therefore$ $5r \leq q$
Since G is a connected planar graph, $r = q+2-p$ by Euler's Planarity Theorem $\therefore$ $5(q+2-p) \leq q$. Hence

$$5q + 10 - 5p \leq q \text{ and so } 4q \leq 5p - 10 = 5(p-2).$$

END