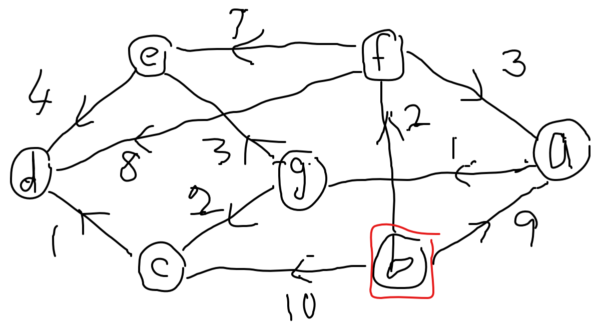
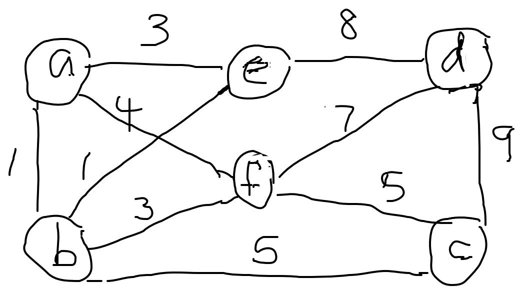
MAD 3305 - GRAPH THEORY FLORIDA INT'L UNIV.
TEST #1 - SPRING 2021 TIME: 75 min.

Answer *all 6* questions. ***No calculators, notes, or on-line data are allowed.*** *An unjustified answer will receive little or no credit.* ***Draw a line to separate each of the 6 solutions to the 6 questions.***

(15) 1. Find the *distances* from b to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2 (a) Find a *graph* with degree sequence ⟨5,4,3,2,2,2⟩ by using the *Graphical Sequence Algorithm.*
(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* and starting at *d.*



(20) 3 (a) Find the *tree* that corresponds to the sequence ⟨3, 5, 1, 3⟩ via the *Prufer's Tree Decoding Algorithm.*
(b) The five characters *a, b, c, d, e* occur with frequencies *4, 12, 5, 20, 9;* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4 (a) Explain what is the difference between a *rooted tree R* and a *spanning tree T* of a graph $G = \langle V, E \rangle$.
(b) Prove that in any tree $T = \langle V(T), E(T) \rangle$, $|E(T)| = |V(T)| - 1$.

(15) 5 (a) Define what is the *distance, d(u,v),* from *u* to *v* in a *weighted digraph G.*
(b) If G is a *disconnected graph* with *p vertices,* prove $|E(G)| \le (p-1).(p-2)/2$. If $|E(G)| = (p-1).(p-2)/2$, is it *always true* that G will be *connected* ?

(15) 6 (a) Explain what is the difference between a *legal flow f* and the ***value*** *of a legal flow f* in a network $N = \langle G, s, t, c \rangle$.
(b) Let T be a *non-trivial* tree and $\{v_1, v_2, v_3, \ldots, v_k\}$ be the set of all the vertices in T with degree ≥ 3. Prove that *no. of leaves in T* $= 2(1-k) + \Sigma_{i=1 \text{ to } k} \deg(v_i)$.
[*You may use* ***any theorems that were proved in class*** *to answer question #6.*]

MAD 3305 – Graph Theory   Florida International Univ.
Solutions to Test # 1   Spring 2021   p (1)

1.

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	$v_i \to \ldots$
∞	$\underline{0}$	∞	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	$b \to a,c,f$
9	.	10	∞	∞	$\underline{2}$	∞	$\{a,c,d,e,f,g\}$	1	$f \to a,d,e$
$\underline{5}$	.	10	10	9	.	∞	$\{a,c,d,e,g\}$	2	$a \to g$
.	.	10	10	9	.	$\underline{6}$	$\{c,d,e,g\}$	3	$g \to e,c$
.	.	$\underline{8}$	10	9	.	.	$\{c,d,e\}$	4	$c \to d$
.	.	.	$\underline{9}$	9	.	.	$\{d,e\}$	5	d
.	.	.	.	$\underline{9}$	.	.	$\{e\}$	6	e
$L(b,\cdot)=$ 5	0	8	9	9	2	6	$\emptyset$		

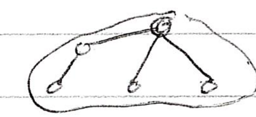
2(a) $\underline{5,4,3,2,2,2}$

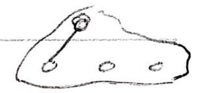
$\underline{3,2,1,1,1}$

$1,0,0,1$

$1,1,0,0$

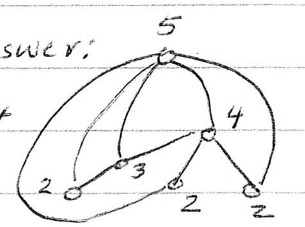
$0,0,0 \longrightarrow$


G_3


G_2


G_1

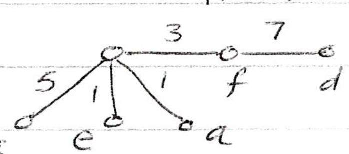
Answer: G_4



(b)

$E(T)$	$V(T)$	a	b	c	d	e	f	i	x_i
$\emptyset$	$\{d\}$	∞	∞	∞	$\underline{0}$	∞	∞	0	d
$\{\overline{df}\}$	$\{d,f\}$	∞	∞	9	.	8	$\underline{7}$	1	f
$\{\overline{df}, \overline{bf}\}$	$\{d,f,b\}$	4	$\underline{3}$	5	.	8	.	2	b
$\{\overline{df}, \overline{bf}, \overline{ab}\}$	$\{d,f,b,a\}$	$\underline{1}$	.	5	.	1	.	3	a
$\{\overline{df}, \overline{bf}, \overline{ab}, \overline{be}\}$	$\{d,f,b,a,e\}$	.	.	5	.	$\underline{1}$	.	4	e
$\{\overline{df}, \overline{bf}, \overline{ab}, \overline{be}, \overline{bc}\}$	$\{a,b,c,d,e,f\}$	.	.	$\underline{5}$	.	.	.	5	c

$T =$



$w(T) = 1+1+3+5+7 = 17$

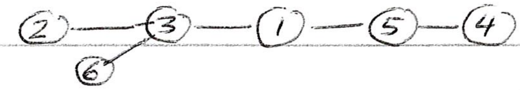
3(a) $p = |\underset{\sim}{s}|+2 =$ length of $(\langle 3,5,1,3\rangle) + 2 = 4+2 = 6$

So the tree will have $V(T) = \{1,2,3,4,5,6\}$.

$d_T(v) = 1 +$ no. of times v appears in $\langle 3,5,1,3\rangle$.

3(a)

i	$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	$l(i)$	$s(i)$
1	2	1	3	1	2	1	2	3
2	2	0	2	1	2	1	4	5
3	2	0	2	0	1	1	5	1
4	1	0	2	0	0	1	1	3
5	0	0	1	0	0	1	3	6

$\therefore$ $T =$ 

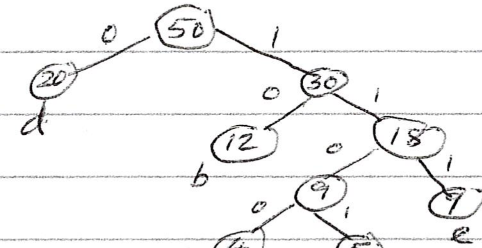
(b) 4, 5, 9, 12, 20

9, 9, 12, 20

12, 18, 20

20, 30

50.



Char	a	b	c	d	e
Freq.	4	12	5	20	9
Code	1100	10	1101	0	111
Code length	4	2	4	1	3

$$\text{W.P.L} = 4(4) + 12(2) + 5(4) + 20(1) + 9(3) = 16 + 24 + 20 + 20 + 27 = 107$$

4(a) A rooted tree is an ordered pair $\langle T, v_0 \rangle$ where T is a tree and v_0 is a distinguished vertex of T. The vertices can then be placed into levels according to their distances from v_0 & T has nothing to do with G. A spanning tree of G is any subgraph H of G such H is a tree & $V(H) = V(T)$. It has no levels.

(b) We will prove the result by induction on $p = |V(T)|$. If $p = 1$, then $T \cong K_1$ and $|E(T)| = 0 = 1 - 1 = |V(T)| - 1$. So the result is true for $p = 1$. Now suppose the result is true for all trees with $\leq p$ vertices. Let T be any tree with $p+1$ vertices. Choose any edge $e \in E(T)$. Then $T - \{e\} = T_1 \cup T_2$ where T_1 & T_2 are disjoint trees with $V(T) = V(T_1) \cup V(T_2)$. So $|E(T)| = |E(T_1)| + |E(T_2)| + |\{e\}| = (|V(T_1)| - 1) + (|V(T_2)| - 1) + 1 = |V(T_1)| + |V(T_2)| - 1 = |V(T)| - 1$. So if the result is true for all T with $\leq p$ vertice, it will be true for all trees with $p+1$ vertices. Hence the result is true for all trees.

5.(a) $d(u,v) = \begin{cases} \text{length of the shortest directed path from } u \text{ to } v \text{ in } G \\ +\infty \quad \text{if there is no directed path from } u \text{ to } v \text{ in } G. \end{cases}$

(b) Suppose G is disconnected. Then we can find two subgraphs G_1 & G_2 of G such that there is no edge from $V(G_1)$ to $V(G_2)$ and $G = G_1 \cup G_2$. Let $|V(G_1)| = k$. Then $|V(G_2)| = |V(G)| - k = p-k$.

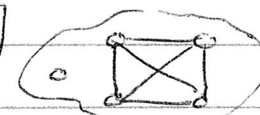
So $|E(G)| \leq k(k-1)/2 + (p-k)(p-k-1)/2$. Hence

$$(p-1)(p-2)/2 - |E(G)| \geq [(p-1)(p-2) - k(k-1) - (p-k)(p-k-1)]/2$$
$$= [(p^2 - 2p - p + 2) - k^2 + k - p^2 + 2pk - k^2 + p - k]/2$$
$$= (-2p + 2 + 2pk - 2k^2)/2 = pk - p - k^2 + 1$$
$$= (k-1)(p-k-1) \geq 0 \quad \text{because } k \geq 1 \text{ \& } p-k \geq 1.$$

$\therefore |E(G)| \leq (p-1)(p-2)/2$.

It is not always true if $|E(G)| = (p-1)(p-2)/2$ that G will be connected.

$\boxed{p=5}$ disconn.



$(p-1)(p-2)/2 = 6$

6(a) A legal flow in N is a function $f: E(G) \to [0,\infty)$ such that for each $e \in E(G)$, $f(e) \leq c(e)$; and $\sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e)$ for each $v \in V(G) - \{s,t\}$. $Val(f) = \sum_{e \in In(t)} f(e) - \sum_{e \in Out(t)} f(e)$.

So the legal flow tells you what is flowing in each edge but the value of the flow is the net amount going into t.

(b) Let $p = V(T)$. Then $|E(T)| = |V(T)| - 1 = p-1$ and Sum of degrees in $T = 2|E(T)| = 2p-2$. Now let l = no. of leaves in T & m = no. of vertices of degree 2.

Then $p = l + m + k$ since $v_1, \ldots, v_k$ are the ones with deg. ≥ 3.

So $l.1 + m.2 + \sum_{i=1}^{k} deg(v_i) = 2p - 2 = 2(l+m+k) - 2$.

$\therefore \sum_{i=1}^{k} deg(v_i) + 2 - 2k = 2l - l + 2m - 2m = l$

$\therefore$ No. of leaves in $T = 2 - 2k + \sum_{i=1}^{k} deg(v_i)$

$$= 2(1-k) + \sum_{i=1}^{k} deg(v_i). \text{ END.}$$