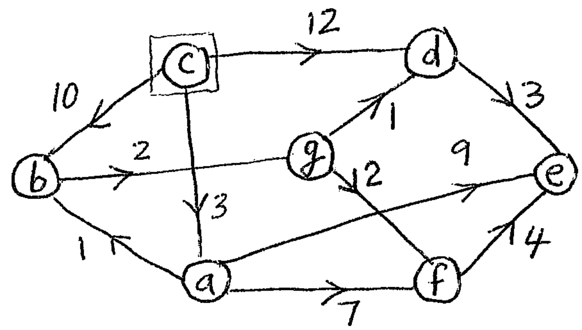
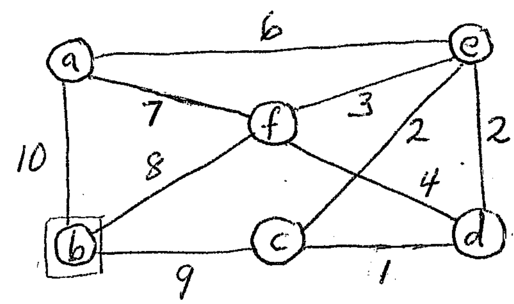
MAD 3301 - GRAPH THEORY | FLORIDA INT'L UNIV.
TEST #1 - SPRING 2023 | TIME: 75 min.

Answer *all 6* questions. ***No calculators, notes, or on-line data are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

(15) 1. Find the *distances* from *c* to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2 (a) Find a *graph* with degree sequence ⟨5,3,3,2,2,2,1⟩ by using the *Graphical Sequence Algorithm.*
(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* and starting at *b.*



(20) 3 (a) Find the *tree* that corresponds to the sequence ⟨4, 3, 1, 4⟩ via the *Prufer's Tree Decoding Algorithm.*
(b) The five characters *a, b, c, d, e* occur with frequencies *12, 6, 10, 4, 16;* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4 (a) Define what is a *walk of length k* in a graph $G = \langle V, E\rangle$.
(b) Let *G* be a graph with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Prove that the number of walks of length *k* from v_i to v_j is $(A^k)[i, j]$ where *A* is the *adjacency matrix* of *G.*

(15) 5 (a) Define what is the *distance, d(u,v),* from *u* to *v* in a *weighted digraph G.*
(b) If G is a *disconnected graph,* prove its complement G^c, must be *connected.* Is it *possible* for both *G* and G^c to be *connected*?

(15) 6 (a) Define what is a *rooted tree* and define what is the *height* of a rooted tree.
(b) Let *T* be a *4-ary* tree with *p* vertices. Prove that $h(T) \geq (1/2).\log_2\{(3p+1)/4\}$.

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Solutions to Test #1 — SPRING 2023

1.

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	$v_i \to$
∞	∞	$\underline{0}$	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	$c \to a,b,d$
$\underline{3}$	10	.	12	∞	∞	∞	$\{a,b,d,e,f,g\}$	1	$a \to b,e,f$
.	$\underline{4}$	.	12	12	10	∞	$\{b,d,e,f,g\}$	2	$b \to g$
.	.	.	12	12	10	$\underline{6}$	$\{d,e,f,g\}$	3	$g \to d,f$
.	.	.	$\underline{7}$	12	8	.	$\{d,e,f\}$	4	$d \to e$
.	.	.	.	10	$\underline{8}$	.	$\{e,f\}$	5	$f \to e$
.	.	.	.	$\underline{10}$	.	.	$\{e\}$	6	$e \to \emptyset$
.	.	.	.	.	.	.	$\emptyset$ STOP		

$d(c,\cdot) = 3 \quad 4 \quad 0 \quad 7 \quad 10 \quad 8 \quad 6.$

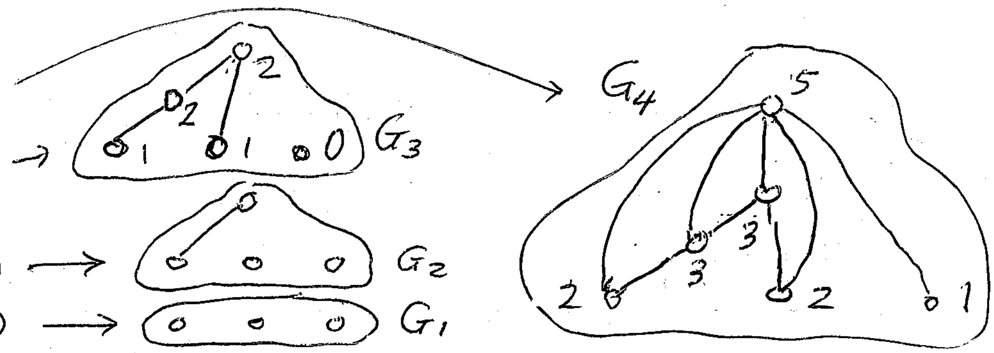
2 (a) 5, 3, $\underline{3, 2, 2, 1}$

2, $\underline{2, 1, 1}$, 0 → G_3

1, 0, 1, 0

1, $\underline{1}$, 0, 0 → G_2

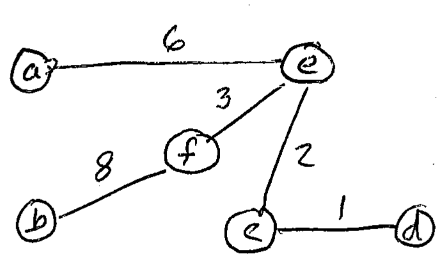
0, 0, 0 → G_1



(b)

$E(T)$	$V(T)$	a	b	c	d	e	f	i	x_0
$\emptyset$	$\{b\}$	∞	0	∞	∞	∞	∞	0	b
$\{\overline{bf}\}$	$\{b,f\}$	10	.	9	∞	∞	$\underline{8}$	1	f
$\{\overline{bf}, \overline{ef}\}$	$\{b,f,e\}$	7	.	9	4	$\underline{3}$	.	2	e
$\{\overline{bf}, \overline{ef}, \overline{ec}\}$	$\{b,f,e,c\}$	6	.	$\underline{2}$	2	.	.	3	c
$\{\overline{bf}, \overline{ef}, \overline{ec}, \overline{dc}\}$	$\{b,c,d,e,f\}$	6	.	.	$\underline{1}$	.	.	4	d
$\{\overline{ae}, \overline{bf}, \overline{ef}, \overline{ec}, \overline{dc}\}$	$\{a,b,c,d,e,f\}$	$\underline{6}$	.	.	.	.	.	5	a

T_{min} =



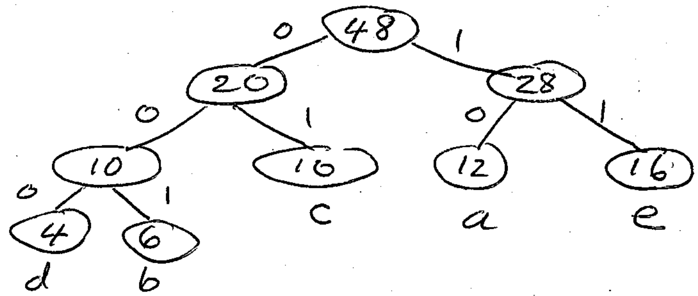
$\omega(T_{min}) = 1 + 2 + 3 + 6 + 8$
$= 20.$

3 (a) $|S| = 4 \Rightarrow p-2=4 \Rightarrow p=6$.

i	$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	$\ell(i)$ — $s(i)$
1	2	1	2	3	1	1	2 — 4
2	2	0	2	2	1	1	5 — 3
3	2	0	1	2	0	1	3 — 1
4	1	0	0	2	0	1	1 — 4
5	0	0	0	1	0	1	plus 4 — 6

T = 

(b)
d, b, c, a, e
4, 6, 10, 12, 16
→ 10, 10, 12, 16
12, 16, → 20
20, → 28
→ 48



Char	a	b	c	d	e
Freq	12	6	10	4	16
Code	10	001	01	000	11
Code length	2	3	2	3	2

W.P.L $= 12(2) + 6(3) + 10(2) + 4(3) + 16(2)$
$= 24 + 18 + 20 + 12 + 32$
$= \boxed{106}$.

4(a) A <u>walk</u> of <u>length k</u> from v_i to v_j in G is an alternating sequence $\langle u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k \rangle$ of vertices & edges of G such that $u_0 = v_i$, $u_k = v_j$, and endpoints$(e_i) = \{u_{i-1}, u_i\}$.

(b) We will prove the result by induction on k. If $k=1$, then no. of walks of length 1 from v_i to v_j in G = no. of edges from v_i to v_j in G. $= A^1[i,j]$ by definition of A. So the result is true for $k=1$.

Now suppose the result is true for k. Then no. of walks of length k in $G = (A^k)[i,j]$. So $\begin{pmatrix}\text{no. of walks of length} \\ k+1 \text{ from } v_i \text{ to } v_j \text{ in } G\end{pmatrix}$

$$= \sum_{\ell=1}^{p} \begin{pmatrix}\text{no. of walks of length } k \\ \text{from } v_i \text{ to } v_\ell \text{ in } G\end{pmatrix} \cdot \begin{pmatrix}\text{no. of walks of length } 1 \\ \text{from } v_\ell \text{ to } v_j \text{ in } G\end{pmatrix}$$

$$= \sum_{\ell=1}^{p} A^k[i,\ell] \cdot A^1[\ell,j] = (A^{k+1})[i,j]$$ by the definition of matrix multiplication.

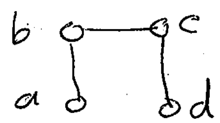
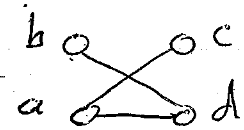
So if the result is true for k, it will be true for $k+1$. By the Princ. of Mathematical Induction, it follows that the result is true for all k. □

5(a) $d(u,v) = \begin{cases} \text{length of the shortest directed path from } u \text{ to } v \text{ in } G \\ +\infty, \text{ if there is no directed path from } u \text{ to } v \text{ in } G. \end{cases}$

(b) Let u & v be any two vertices in G^c. There are 2 cases.

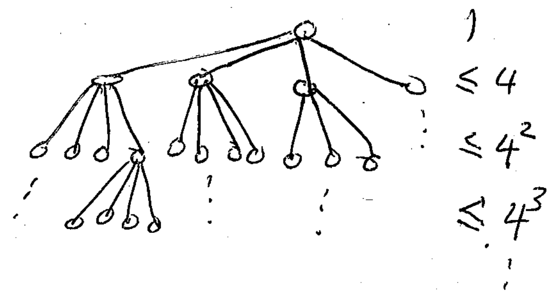
Case (i): $\overline{uv} \notin E(G)$. In this case $\overline{uv} \in E(G^c)$, so we instantly get a path from $u - v$ in G^c.

Case (ii) $\overline{uv} \in E(G)$. In this case u & v belong to the same component of G & since G is disconnected there will be another vertex w in G which is in a different component. But then $\overline{uw} \in E(G^c)$ & $\overline{wv} \in E(G^c)$ and $u-w-v$ will be a path from u to v in G^c. So in either case we found a path from u to v in G^c. Since u & v were arb., G^c is connected.

(c) Yes, G & G^c can both be connected. | Ex. $G =$  $G^c =$ 

6(a) A rooted tree is an ordered pair $\langle T, v_0 \rangle$ where T is a tree and v_0 is a distinguished vertex of T. The vertices of a rooted tree can be classified into levels according to their distances from v_0 & $h(T)$ = the highest levels that exists.

(b) Let $p = |V(T)|$ and $k = h(T)$.



Since T is a 4-ary tree,

level 0 will have 1 vertex

level 1 " " ≤ 4 vertices

level 2 " " $\leq 4^2$ vertices

and in general level i will have $\leq 4^i$ vertices

So $p \leq 1 + 4 + 4^2 + \dots + 4^k = (4^{k+1} - 1)/(4-1)$

$= (4^{k+1} - 1)/3$

$\therefore$ $3p \leq 4^{k+1} - 1$, and so $3p + 1 \leq 4^{k+1}$ $\therefore$ $\frac{3p+1}{4} \leq 4^k = 2^{2k}$

Hence $2^{2k} \geq \frac{3p+1}{4}$ & so $2k \geq \log_2\left(\frac{3p+1}{4}\right)$.

Thus $h(T) = k \geq \frac{1}{2} \log_2\left(\frac{3p+1}{4}\right)$.

END