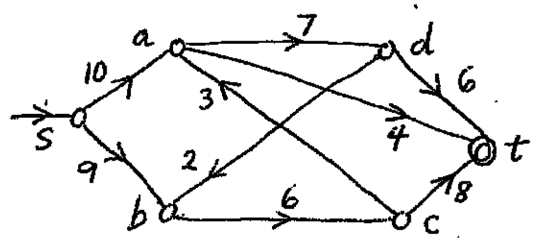
MAD 3301 - GRAPH THEORY
TEST #2 - SPRING 2023

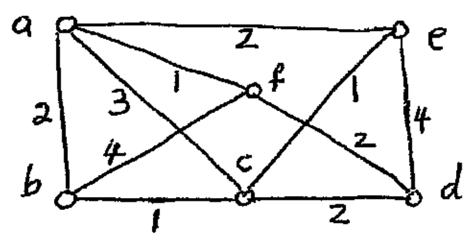
FLORIDA INT'L UNIV.
TIME: 75 min.

Answer all 6 questions. No Calculators or Cellphones are allowed. An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

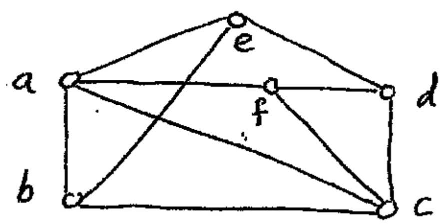
(15) 1. Find a maximal flow f^* in the network on the right by using the *Ford-Fulkerson Algorithm.* Also find the source-separating set of vertices S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

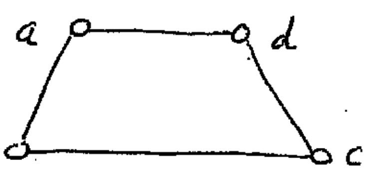


(18) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm.* [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm.*

(b) Prove that $\chi(G) \leq 2$ if G has only even cycles (i.e., if G has no odd cycles).



(15) 5 (a) Define what is a *minimum postman walk* and define what is a *minimum salesman walk* in a weighted multi-graph G.

(b) Write down *Ore's theorem* & use it to prove that any graph G with $\deg(x) + \deg(y) \geq p-1$ for all pairs of non-adjacent vertices x &y, has a *Hamilton path.*

(15) (a) Define what is a *planar* graph G & define what is the *dual* of *G with respect to a planar embedding* $\mathcal{E}$ of G.

(b) Let $\mathcal{E}$ be a planar embedding of a connected planar-graph G in which each region is bounded by at least 7 edges. Prove that $5q \leq 7(p-2)$.
[*You may use any theorem that was proved in class for Qu.#6, if needed.*]

MAD 3301 - Graph Theory — Florida Int'l Univ.

Solutions to Test #2 — Spring 2023

#1 1st Aug. semi-path

$s \xrightarrow{(0,10)} a \xrightarrow{(0,4)} t$

slacks: 10 4 $(\mu_1 = 4)$

2nd Aug. semi-path

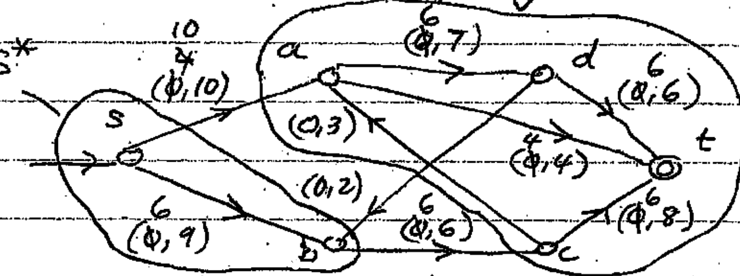
$s \xrightarrow{(4,10)} a \xrightarrow{(0,7)} d \xrightarrow{(0,6)} t$

slacks 6 7 6 $(\mu_2 = 6)$

3rd aug. semipath

$s \xrightarrow{(0,9)} b \xrightarrow{(0,6)} c \xrightarrow{(0,8)} t$

slacks 9 6 8 $(\mu_3 = 6)$

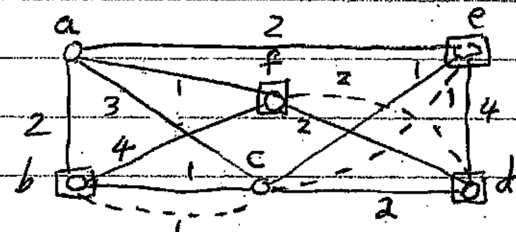


$S^* = \{u \in V(G)\text{: there is an aug. semipath from } s \text{ to } u\} = \{s, b\}$. $c[Cut(S^*)] = 10 + 6 = 16$.

$Val(f^*) = \text{net flow into } t = 6 + 4 + 6 = 16$ ✓

#2

	b	d	e	f
b	.	3	2	3
d	.	.	3	2
e	.	.	.	3
f	.	.	.	.



$\{b,d\} + \{e,f\}$: $3 + 3 = 6$ $\{b,e\} + \{d,f\}$: $2 + 2 = 4$ ✓ $\{b,f\} + \{d,e\}$: $3 + 3 = 6$

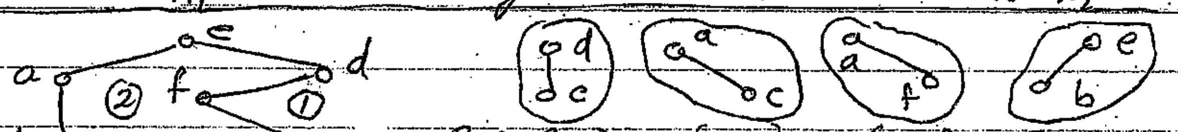
Minimum Postman walk:

$a \xrightarrow{2} e \xrightarrow{4} d \xrightarrow{2} f \xrightarrow{2} d \xrightarrow{2} c \xrightarrow{1} e \xrightarrow{1} c \xrightarrow{1} b \xrightarrow{1} c \xrightarrow{3} a \xrightarrow{1} f \xrightarrow{4} b \xrightarrow{2} a$.

Length of walk $= 26$. Check: $w(G) + 4 = 22 + 4 = 26$ ✓

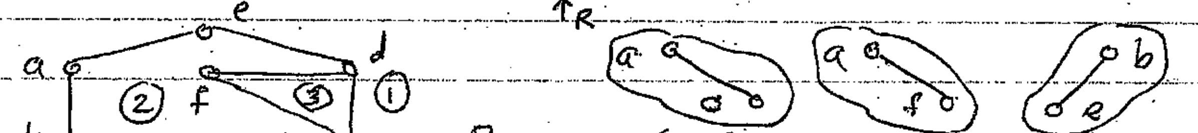
#3. i H_i Segments of G relative to H_i

1



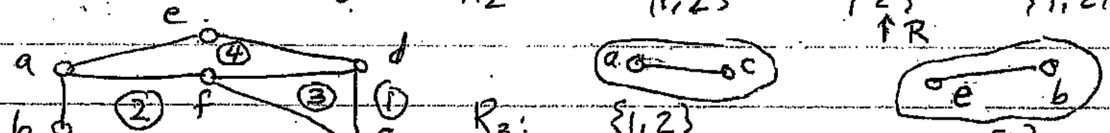
R_1: {1,2} ↑R {1,2} {1,2} {1,2}

2



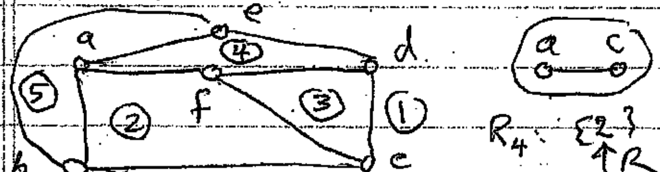
R_2: {1,2} {2} ↑R {1,2}

3



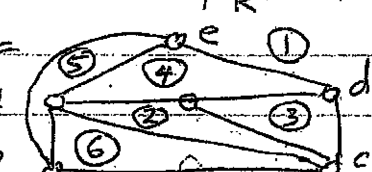
R_3: {1,2} {1} ↑R

4



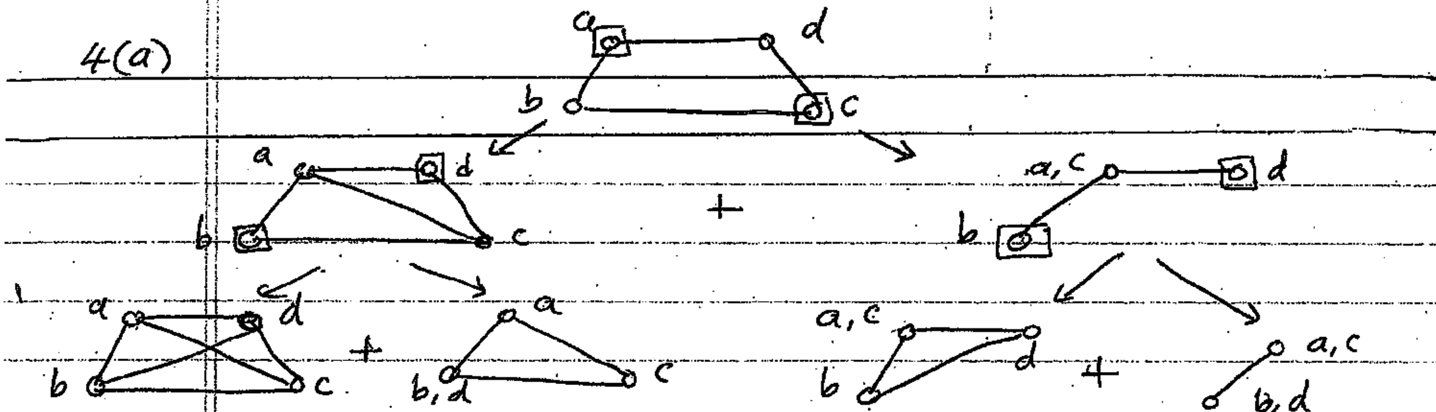
R_4: {2} ↑R

$H_5 =$



G is PLANAR

4(a)



$\therefore\ P_G(\lambda) = P_{K_4}(\lambda) + 2P_{K_3}(\lambda) + P_{K_2}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 2\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)$

$= \lambda(\lambda-1)[(\lambda-2)(\lambda-3) + 2(\lambda-2) + 1] = \lambda(\lambda-1)[\lambda^2 - 3\lambda + 3]$.

(b) Let $\mathcal{F} = T_1 \cup T_2 \cup \dots \cup T_k$ be a spanning forest of G. Choose a vertex v_i in T_i and designate it as the root of T_i, so that we get k rooted trees $\langle T_i, v_i \rangle$. Now color the even levels of each tree T_i with color #1 & the odd levels with color #2. Then add back the edges taken out of G (to form $\mathcal{F}$) one at a time. Each time we add an edge it must join two vertices of different colors – otherwise we would get an odd cycle in G. So the coloring is legal for G. $\therefore \chi(G) \le 2$.

5(a) A minimum postman walk of G is a closed walk which includes each edge of G at least once and is of smallest possible total length. A minimum salesman walk of G is a closed walk which includes each vertex of G at least once and is of smallest possible total length.

(b) Ore's theorem: If G is any graph with p vertices, $p \ge 3$, and for any pair of non-adjacent vertices x & y, $\deg(x) + \deg(y) \ge p$, then G has a Hamilton cycle.

Suppose G is a graph with $\deg(x) + \deg(y) \ge p-1$ for all pairs of non-adjacent vertices x & y. Now if $p = 1$, then $G \cong K_1$ & the empty path is a Hamilton path of G. So we may assume that $p \ge 2$. Let H be the graph obtained from G by adding a new vertex v_{p+1} and edges from v_{p+1} to each of the vertices in G. Then $\deg_H(x) + \deg_H(y) \ge (p-1) + 2 = p+1$ for all pairs of non-adjacent vertices in H. Since H hase has $p+1$ vertices, it follows from Ore's Theorem that H has a Hamilton cycle C. Now if we remove the vertex v_{p+1} from C, we will get a Hamilton path of G.

6(a) A graph G is *planar* if we can draw it in the plane so that no two edges intersect except possibly at their endpoints. The dual of G with respect to the embedding $\mathcal{E}$ is the *multigraph* $\langle V(G_{\mathcal{E}}^*), E(G_{\mathcal{E}}^*)\rangle$ where $V(G_{\mathcal{E}}^*)$ = set of regions into which $\mathcal{E}$ partitions the plane and two regions $R1$ & $R2$ share an edge for each edge of G that is between $R1$ & $R2$

(b) Let r = number of regions into which $\mathcal{E}$ partitions the plane, p = no. of vertices of G, and q = no. of edges of G. Also let $A_1, A_2, \ldots, A_r$ be the regions of $\mathcal{E}$ and $e(A_i)$ = no. of edges bounding A_i. Then $e(A_i) \geq 7$ for each $i = 1, \ldots, r$.

So $7r = 7 + 7 + \cdots 7$ (r times)

$\leq e(A_1) + e(A_2) + \cdots + e(A_r)$

$\leq 2q$ because each edge of G is in at most two regions.

$\therefore$ $7r \leq 2q$. But $r = q + 2 - p$ because G is a connected planar graph (Euler's Planarity Theorem)

So $7(q+2-p) \leq 2q$. $\therefore$ $7q + 14 - 7p \leq 2q$

$\therefore$ $5q \leq 7p - 14$. Hence $5q \leq 7(p-2)$. END