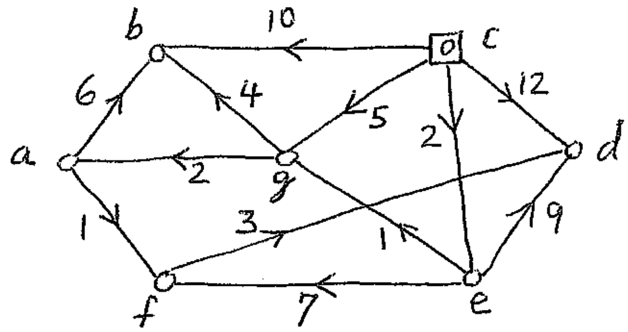
MAD 3305 - GRAPH THEORY
TEST #1 - SPRING 2024

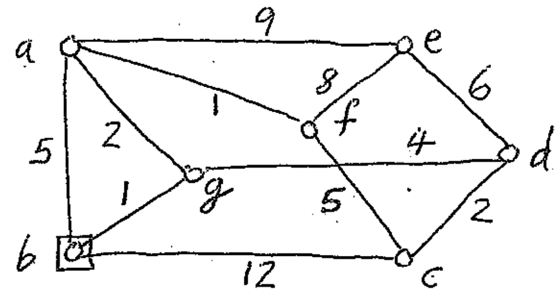
FLORIDA INT'L UNIV.
TIME: 75 min.

Answer *all 6* questions. ***No calculators, cell-phones , or notes are allowed.*** *An unjustified answer will receive little or no credit.* BEGIN EACH OF THE 6 QUESTION ON 6 SEPARATE PAGES.

(15) 1. Find the *distances* from *c* to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2 (a) Find a *graph* with degree sequence ⟨5,4,3,3,2,1⟩ by using the *Graphical Sequence Algorithm.*
(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* and starting at *b*.



(20) 3 (a) Find the *tree* that corresponds to the sequence ⟨5, 3, 1, 5⟩ via the *Prufer's Tree Decoding Algorithm.*
(b) The five characters *a, b, c, d, e* occur with frequencies *5, 11, 6, 20, 8;* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4 (a) Define what is the ***adjacency matrix*** of a **digraph** G with V(G)={1,2,3, ...,p}.
(b) Prove that $(A^k)[i, j]$ = number of ***directed walks*** of length k from i to j in G.

(15) 5 (a) Define what is the ***distance***, *d(u,v),* from *u* to *v* in a *weighted digraph G.*
(b) If G is a ***disconnected graph*** with *p vertices*, prove $|E(G)| \leq (p-1)(p-2)/2$.
If $|E(G)| < (p-1)(p-2)/2$, is it possible for G to be ***connected***?

(15) 6 (a) Define what is a ***legal flow*** f and the ***value*** *of a legal flow* f in a *network* $N = \langle G, s, t, c\rangle$.
(b) Let T be a *non-trivial* tree & $\{v_1, v_2, v_3, \ldots, v_n\}$ be the set of all the vertices in T with degree ≥ 3. Prove that *no. of leaves in T* $= (2-2n) + \Sigma_{i=1 \text{ to } n} \deg(v_i)$.

[*You may use* ***any theorems proved in class*** *to answer question #6.*]

MAD 3301 ~ Graph Theory — Florida Int'l Univ.

Solutions to Test #1 — Spring 2021

1.

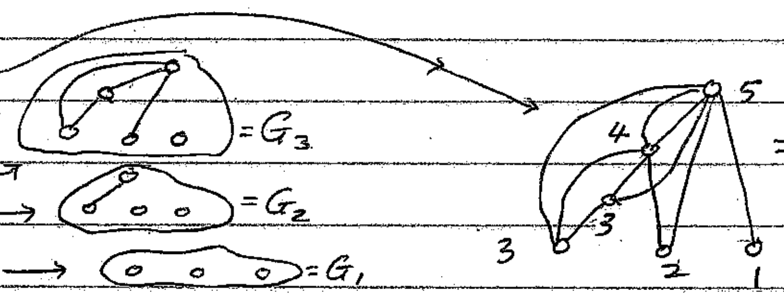
$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	T	i	$v_0(i)$
∞	∞	0	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	$c \to b,d,e,g$
∞	10	·	12	2	∞	5	$\{a,b,d,e,f,g\}$	1	$e \to d,f,g$
∞	10	·	11	·	9	3	$\{a,b,d,f,g\}$	2	$g \to a,b$
5	7	·	11	·	9	·	$\{a,b,d,f\}$	3	$a \to b,f$
·	7	·	11	·	6	·	$\{b,d,f\}$	4	$f \to d$
·	7	·	9	·	·	·	$\{b,d\}$	5	b
·	·	·	9	·	·	·	$\{d\}$	6	d
$d(c,\cdot)=$ 5	7	0	9	2	6	3	$\emptyset$ STOP		

2.(a) 5, 4, 3, 3, 2, 1

3, 2, 2, 1, 0

1, 1, 0, 0 →

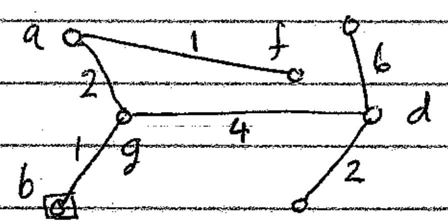
0, 0, 0 →



(b)

$E(T)$	$V(T)$	a	b	c	d	e	f	g	i	$x_0(i)$
$\emptyset$	$\{b\}$	∞	0	∞	∞	∞	∞	∞	0	b
$\{\overline{bg}\}$	$\{b,g\}$	5	·	12	∞	∞	∞	1	1	g
$\{\overline{bg},\overline{ga}\}$	$\{a,b,g\}$	2	·	12	4	∞	∞	·	2	a
$\{\overline{bg},\overline{ga},\overline{af}\}$	$\{a,b,f,g\}$	·	·	12	4	9	1	·	3	f
$\{\overline{bg},\overline{ga},\overline{af},\overline{gd}\}$	$\{a,b,d,f,g\}$	·	·	5	4	8	·	·	4	d
$\{\overline{bg},\overline{ga},\overline{af},\overline{gd},\overline{dc}\}$	$\{a,b,c,d,f,g\}$	·	·	2	·	6	·	·	5	c
$\{\overline{bg},\overline{ga},\overline{af},\overline{gd},\overline{dc},\overline{de}\}$	$\{a,b,c,d,e,f\}$	·	·	·	·	6	·	·	6	e

$T_{min} =$

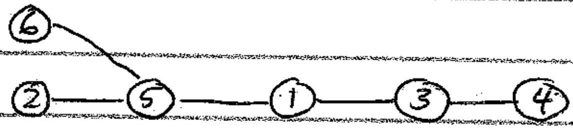


$\omega(T_{min}) = 1+2+1+4+2+6$

$= \boxed{16}$

3 (a) $\underset{\sim}{S} = \langle 5,3,1,3 \rangle$, so $p = |\underset{\sim}{S}| + 2 = 4 + 2 = \boxed{6}$

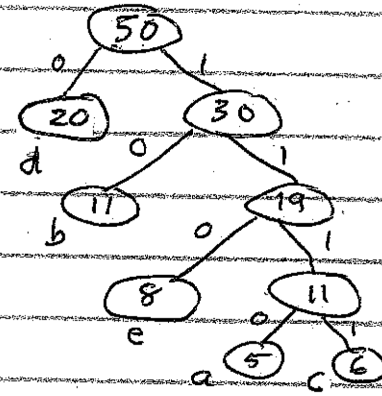
i	$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	$\ell(i)$ — $s(i)$
1	2	1	2	1	3	1	2 — 5
2	2	0	2	1	2	1	4 — 3
3	2	0	1	0	2	1	3 — 1
4	1	0	0	0	2	1	1 — 5
5	0	0	0	0	1	1	plus 5 — 6

$T =$ 

(b)

Char.	a	b	c	d	e
Freq.	5	11	6	20	8
Code	1110	10	1111	0	110
Code length	4	2	4	1	3

5, 6, 8, 11, 20
8, 11, 11, 20
11, 19, 20
20, 30
50



WPL (coding) $= 5(4) + 11(2) + 6(4) + 20(1) + 8(3) = \boxed{110}$

4 (a) The adjacency matrix A_G of a digraph with vertices $\{1, 2, 3, \dots, p\}$ is the $p \times p$ matrix defined by $A_G[i,j] =$ no. of directed edges from i to j in G.

(b) We will prove the result by induction on k. If $k=1$, then the no. of directed walks of length 1 from i to j in G = no. of dir. edges from i to j in G. So the result is true for $k=1$. Now suppose the result is true for k. Then no. of directed walks of length k from any i to any j $= (A^k)[i,j]$.

So $\begin{pmatrix}\text{no. of directed walks} \\ \text{of length } k+1 \text{ from } i \text{ to } j \text{ in } G\end{pmatrix} = \sum_{m=1}^{p} \begin{pmatrix}\text{no. of dir. walks of} \\ \text{length } k \text{ from } i \text{ to } m\end{pmatrix} \cdot \begin{pmatrix}\text{no. of dir. walks of} \\ \text{length 1 from } m \text{ to } j\end{pmatrix}$

$= \sum_{m=1}^{p} A^k[i,m] \cdot A^1[m,j] = (A^{k+1})[i,j]$ by the def. of matrix mult.

So if the result is true for k, it will be true for $k+1$. By the Principle of Mathematical Induction, it follows that the result is true for all k.

5 (a) $d(u,v) = \begin{cases} \text{length of the shortest } \underline{\text{directed}} \text{ path from } u \text{ to } v \text{ in } G \quad \& \\ +\infty, \text{ if there is no directed path from } u \text{ to } v \text{ in } G. \end{cases}$

5(b) Suppose G is a disconnected graph with p vertices. Then we can split G into two graphs G_1 & G_2 such that there is no edge from G_1 to G_2. Let $|V(G_1)| = k$. Then $|V(G_2)| = p-k$. So

$$|E(G)| = |E(G_1)| + |E(G_2)| \leq \frac{k(k-1)}{2} + \frac{(p-k)(p-k-1)}{2}$$

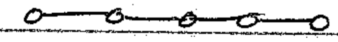
$$\therefore \frac{(p-1)(p-2)}{2} - |E(G)| \geq \frac{(p-1)(p-2) - k(k-1) - (p-k)(p-k-1)}{2}$$

$$= \frac{(p^2-3p+2) - k^2+k - p^2+2pk-k^2+p+pk}{2}$$

$$= (-2p+2+2pk-2k^2)/2$$

$$= kp+1-p-k^2 = (k-1)(p-k-1) \geq 0$$

since $k-1 \geq 0$ & $(p-k)-1 \geq 0$. Hence $|E(G)| \leq \frac{(p-1)(p-2)}{2}$.

(c) Yes it is possible for $|E(G)|$ to be less than $(p-1)(p-2)/2$ and still have G connected. Just take $p=5$ and $G =$ . Then $|E(G)| = 4 < 6 = (5-1)(5-2)/2$

6 (a) A legal flow in a network $N = \langle G, c, s, t\rangle$ is any function $f: E(G) \to [0,\infty)$ such that $f(e) \leq c(e)$ for each $e \in E(G)$ and $\sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e)$ for each $v \in V(G) - \{s,t\}$.

The value of the flow f is defined by $Val(f) = \sum_{e \in In(t)} f(e) - \sum_{e \in Out(t)} f(e)$.

(b) Let $p = |V(T)|$, l = no. of leaves in T, & k = no. of vertices of degree 2 in T. Then $p = l+k+n$. Now we know that the tree T has $p-1$ edges and that the sum of the degrees in any graph $= 2$(no. of edges). So

sum of degrees in $T = 1 \cdot l + 2 \cdot k + \sum_{i=1}^{n} deg(v_i)$

will be equal to $2(p-1)$ which is $2(l+k+n-1)$

$$\therefore l + 2k + \sum_{i=1}^{n} deg(v_i) = 2l + 2k + 2n - 2$$

$$\therefore \sum_{i=1}^{n} deg(v_i) = l + (2n-2)$$

$\therefore$ no. of leaves in $T = l = (2-2n) + \sum_{i=1}^{n} deg(v_i)$. END