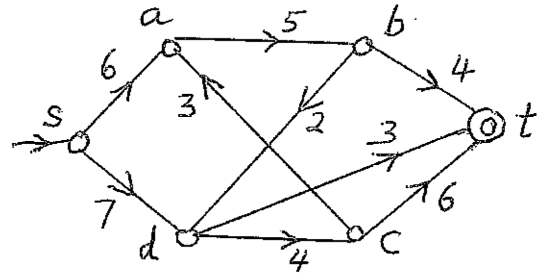
MAD 3301 - GRAPH THEORY FLORIDA INT'L UNIV.

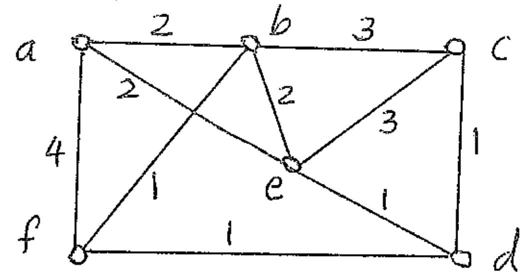
TEST #2 - SPRING 2024 TIME: 75 min.

*Answer **all 6 questions**. **No calculators, notes, or cellphones are allowed.** An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

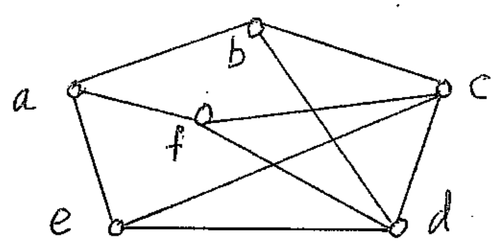
(15) 1. Find a *maximal flow* f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the *source-separating set of vertices* S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

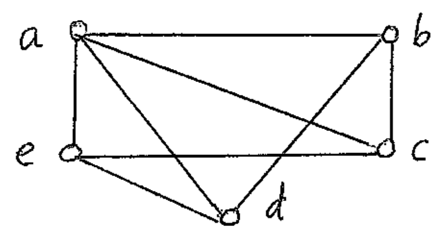


(18) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(22) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm*.

(b) Define the ***chromatic number*** of a graph G & prove for any *non-trivial* tree T, $\chi(T) = 2$.



(15) 5 (a) Define what is a ***minimum postman walk*** and define what is a ***minimum salesman walk*** of a *weighted multi-graph* G.

(b) Write down the *Euler-Circuit theorem* & use it to prove that any connected graph G, with *exactly two odd vertices*, has an *open Euler-trail*.

(15) 6 (a) Define what is a ***planar graph G*** and define what is the ***dual of G*** *with respect to a planar embedding $\mathcal{E}$* of G.

(b) Let $\mathcal{E}$ be a planar embedding of a *connected planar-graph* G in which each region is bounded by *at least 6 edges*. Prove that $2q \leq 3(p-2)$.

[*You may use any theorem that was proved in class for Qu.#6, if needed.*] END

MAD 3301 – Graph Theory — Florida International Univ.

Solutions to Test #2 — Spring 2024

#1. 1st Aug. semi-path

$s \xrightarrow{(0,7)} d \xrightarrow{(0,3)} t$

slacks 7 3 $\mu_1 = 3$

2nd Aug. semi-path

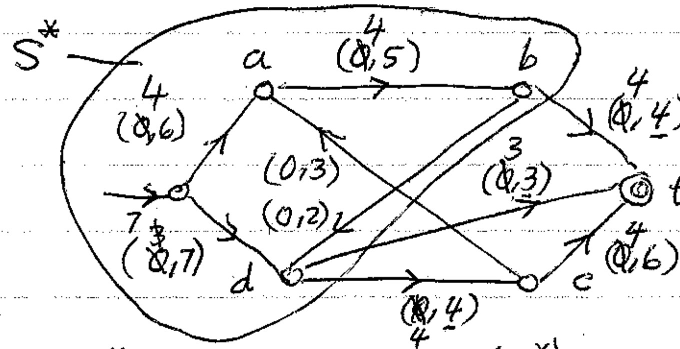
$s \xrightarrow{(3,7)} d \xrightarrow{(0,4)} c \xrightarrow{(0,6)} t$

slacks 4 4 6 $\mu_2 = 4$

3rd Aug. semi-path

$s \xrightarrow{(0,6)} a \xrightarrow{(0,5)} b \xrightarrow{(0,4)} t$

Slacks 6 5 4 $\mu_3 = 4$

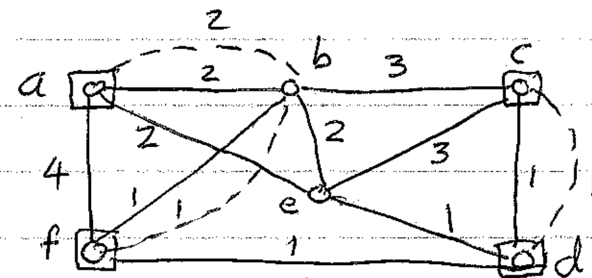


$S^* = \{s,a,b,d\}$ $c(S^*) = 4+3+4 = 11$

$Val(f^*) =$ net flow into $t = 3+4+4 = 11$ ✓

#2.

dist	a	c	d	f
a	.	4	3	3
c	.	.	1	2
d	.	.	.	1
f	.	.	.	.

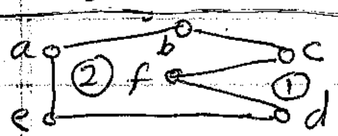


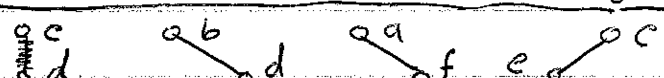
$\{a,c\}+\{d,f\}$: $4+1=5$ $\{a,d\}+\{c,f\}$: $3+2=5$ $\{a,f\}+\{c,d\}$: $3+1=4$ ✓

Minimum postman walk W is:

$a \xrightarrow{2} b \xrightarrow{2} a \xrightarrow{2} e \xrightarrow{2} b \xrightarrow{1} f \xrightarrow{1} b \xrightarrow{3} c \xrightarrow{3} e \xrightarrow{1} d \xrightarrow{1} c \xrightarrow{1} d \xrightarrow{1} f \xrightarrow{4} a$ length(W) = 24.

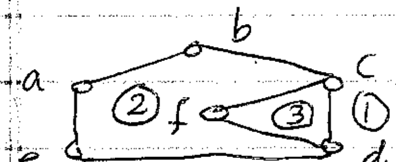
#3 i H_i Segments of G relative to H_i

1 



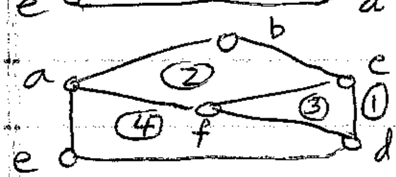
$R_1(S)$: {1,2} {1,2} {1,2} {1,2}

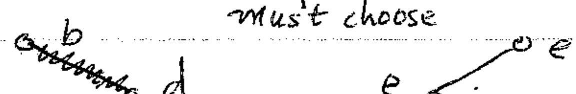
choose

2 

$R_2(S)$: {1,2} {2} {1,2}

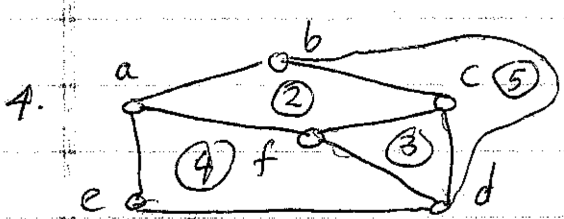
must choose

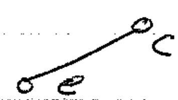
3. 



$R_3(S)$: {1} {1}

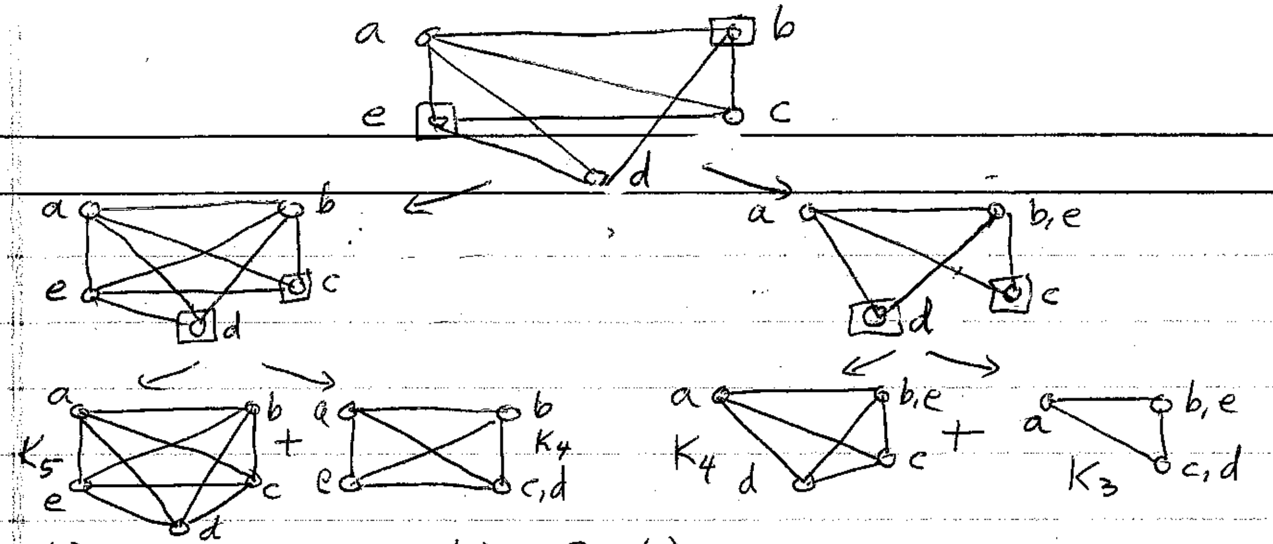
choose

4. 



$R_4(S) = \emptyset$ → ∴ G is NON-PLANAR

4(a)



$$\therefore\ P_G(\lambda) = P_{K_5}(\lambda) + 2P_{K_4}(\lambda) + P_{K_3}(\lambda)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1] = \lambda(\lambda-1)(\lambda-2)[\lambda^2-5\lambda+7$$

(b) The chromatic number, $\chi(G)$ = the smallest numbers of colors that can be use to legally color G (i.e., so that adj. vertices have diff. colors. If T is a non-trivial tree, then T has at least one edge — and since we need 2 colors for the endpoints of that edge, $\chi(T) \geq 2$. Now select any vertex v in T & designate v as the root. Then $\langle T, v\rangle$ will be a rooted tree and the vertices of T will be split into levels. Color the even levels with color #1 & the odd levels with color #2. This will give us a legal coloring of T because no two vertices in the even levels share an edge & no two vertices in the odd levels share an edge. So $\chi(T) \leq 2$. Since $\chi(T) \geq 2$, we get $\chi(T) = 2$.

5(a) A <u>minimum postman walk</u> of G is any <u>closed</u> walk of G which includes each <u>edge</u> of G (at least once) and is of the smallest possible total length. A <u>minimum salesman</u> walk of G is a <u>closed</u> walk of G which includes each vertex of G (at least once) & is of the smallest possible total length.

(b) <u>Euler - circuit Theorem</u>: A <u>connected</u> multi-graph G has an Euler-circuit $\Leftrightarrow$ each vertex of G is of even degree.
[An Euler-circuit of G is a <u>closed</u> walk of G which includes every edge of G exactly <u>once</u>.]

5(b) Let G be a connected graph with exactly two vertices of odd degree — call these odd vertices u & v. Put $H = G \cup \{\overline{uv}\}$. Then H will be a connected multi-graph with all vertices of even degree. So by the Euler-circuit Theorem, H will have an Euler circuit, Q say. Now if we delete the edge $\overline{uv}$ from this Euler circuit Q, we will get an open Euler-Walk from u to v in G. Hence G will have an open Euler-trail (because no edge will be repeated since we took out the only possible repeated edge)

#6 (a) A graph G is planar if we can draw it in the plane so that no two edges intersect, except possible at their end-points. The dual G^* of a planar-graph G w.r.t. the planar embedding $\mathcal{E}$ is the graph $G^* = \langle V(G^*), E(G^*) \rangle$ where $V(G^*)$ = the set of regions into which $\mathcal{E}$ partitions $\mathbb{R}^2$ and for each edge that two regions R_1 & R_2 share in $\mathcal{E}$, we get a a corresponding edge between R_1 & R_2 in $E(G^*)$.

(b) Let $A_1, A_2, \ldots, A_r$ be the regions into which the plane $\mathbb{R}^2$ is partitioned by $\mathcal{E}$. Then if $e(A_i)$ = no. of edges of A_i,

$$6.r \leq e(A_1) + e(A_2) + \cdots + e(A_r) \leq 2q$$ because $e(A_i) \geq 6$

and each edge is counted as a boundary at most twice. So $6r \leq 2q$ & thus $3r \leq q$. But G is a connected planar graph, so $r = q+2-p$ by Euler's Planarity Theorem. Hence

$3(q+2-p) \leq q$. $\therefore$ $3q - q \leq 3p-6$ So $2q \leq 3p-6$.

END