# CHAPTER 1

1.1 First of all there is a misprint in this problem - the two graphs are in Fig. 1.1.5. Define $\varphi_1$ by

$\varphi_1(a) = x$ $\qquad \varphi_1(b) = r$

$\varphi_1(c) = y$ $\qquad \varphi_1(d) = s$

$\varphi_1(e) = z$ $\qquad \varphi_1(f) = t$

Then $\varphi_1$ is an isomorphism. If we define $\varphi_2$ by

$\varphi_2(a) = r$ $\qquad \varphi_2(b) = x$

$\varphi_2(c) = s$ $\qquad \varphi_2(d) = y$

$\varphi_2(e) = t$ $\qquad \varphi_2(f) = z$,

then $\varphi_2$ will also be an isomorphism.
There are 70 other isomorphisms between the two graphs $G_1$ and $G_2$

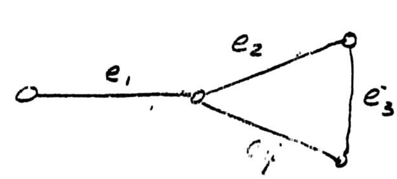
1.4 Let $\varphi : V(G_1 \times G_2) \rightarrow V(G_2 \times G_1)$ be defined by

$$\varphi(\langle v_1, w_1 \rangle) = \langle w_1, v_1 \rangle$$

Then it is easy to check that $\varphi$ is an isomorphism.

1.5 The result analogous to theorem 1.3.1 is
"If $A$ is the adjacency matrix of a digraph $D$ with vertices $v_1, \ldots, v_p$, then the number of walks of length $n$ <u>from</u> $v_i$ to $v_j$ is $A^n[i,j]$."

1.6 Consider the graph on the right




$e_1, e_2, e_3, e_4, e_1$ is a walk but not a trail

$e_1, e_2, e_3, e_4$ is a trail but not a path.

1.7 Order of $G_1 \cup G_2 = |V(G_1) \cup V(G_2)| \leq p_1 + p_2$

Size of $G_1 \cup G_2 = |E(G_1) \cup E(G_2)| \leq q_1 + q_2$

Order of $G_1 \times G_2 = |V(G_1)| \cdot |V(G_2)| = p_1 \cdot p_2$

Size of $G_1 \times G_2 = |V(G_1)| \cdot |E(G_2)| + |V(G_2)| \cdot |E(G_1)|$

$= p_1 \cdot q_2 + p_2 \cdot q_1$

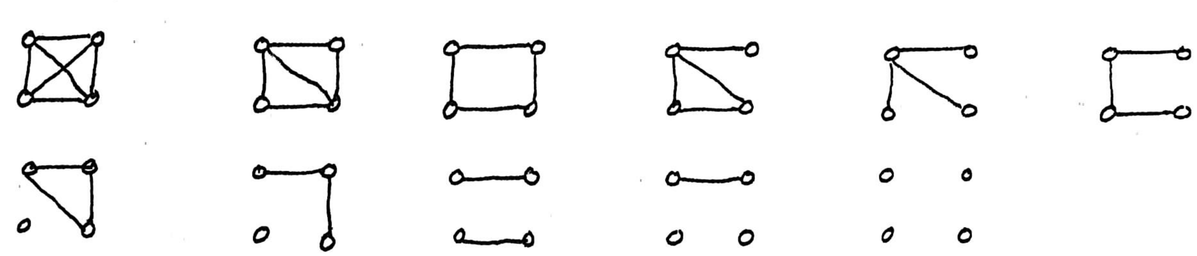
Order of $\bar{G}_1 = p_1$

Size of $\bar{G}_1 = [p_1(p_1 - 1)/2] - q_1$

1.8 For a counter example look at $P_2[P_3]$ and $P_3[P_2]$ on page 13.

1.9 Hint: Use the fact that there is a one to one correspondence between the vertices and that if two vertices are joined by an edge in one graph, then the corresponding two vertices will be joined by an edge in the other graph.

1.10 There are 11 different non-isomorphic graphs on 4 vertices



1.12 Method 1: Each vertex has $p-1$ edges emanating from it. So each vertex has degree $p-1$. So the total degree sum is $p(p-1)$. So no. of edges $= p(p-1)/2$.

Method 2: An edge is just an unordered pair of distinct vertices. There are $\binom{p}{2} = \frac{p(p-1)}{2}$ unordered pairs (from combinatorics)

1.14 Since $G \cong \bar{G}$, we must have $|E(G)| = |E(\bar{G})|$ (i.e. $G$ and $\bar{G}$ must have the same number of edges). Also since $\bar{G}$ is the complement of $G$, $|E(G)| \;\; |E(\bar{G})| = |E(K_p)| = p(p-1)/2$
So $|E(G)| = p(p-1)/4$. Now there are two cases

<u>Case (i)</u>: $p$ is even.
In this case $p-1$ is odd, so since $|E(G)|$ is an integer, 4 must divide $p$. $\therefore$ $p \equiv 0 \pmod 4$

<u>Case (ii)</u>: $p$ is odd
In this case $p-1$ must be even, and again since $|E(G)|$ is an integer and $p$ is odd, 4 must divide $p-1$. So $p-1 \equiv 0 \bmod 4$. $\therefore$ $p \equiv 1 \pmod 4$

1.16 Let $V_1$ and $V_2$ be the two partite sets of the bipartite graph. Then

$|V_1|.d =$ no. of edges in $G$ and
$|V_2|.d =$ no. of edges in $G$

So we must have $|V_1| = |V_2|$.

1.17 Again there was a misprint in the problem. You should find all the non-isomorphic digraphs with 3 vertices. There 16 different ones.
(There are 218 non-isomorphic digraphs on 4 vertices)

1.18 $A$ is the adjacency matrix of a digraph if and only if $A$ is a square matrix with 0's and 1's only as entries and all the entries on the main diagonal are 0's

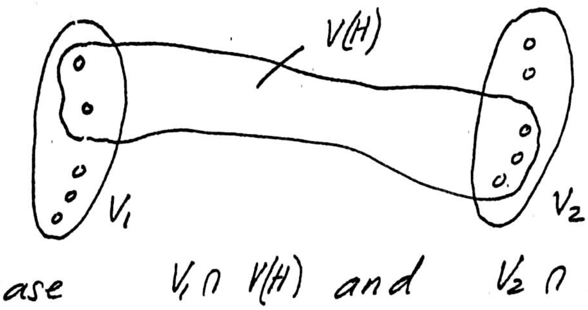
1.19 (a) not graphical, (b) graphical, (c) not graphical

1.20 Suppose $d_1, \ldots, d_p$ is a graphical. Then we can find a graph $G$ with $d_1, \ldots, d_p$ as its degree sequence. Now look at $\bar{G}$. The degree sequence of $\bar{G}$ will be $(p-1)-d_1, (p-1)-d_2, \ldots, (p-1)-d_p$. So $p-d_1-1, p-d_2-1, \ldots, p-d_p-1$ will be graphical

1.23 If we add all the other possible edges to a graph $G$ of order $n$, we will get the complete graph $K_n$. Now if we take out back the edges we added we will get $G$ and this shows that $G$ is a subgraph of $K_n$.

1.24 There is a small error in this problem. The problem should read " Show that every <u>non-trivial</u> subgraph of a bipartite graph is bipartite."
Let $H$ be a non-trivial subgraph of a bipartite graph $G$, and $V_1$ and $V_2$ be the partite sets of vertices in $G$. There are 2 cases

<u>Case (i)</u> : $V(H) \cap V_1 \neq \emptyset$ and $V(H) \cap V_2 \neq \emptyset$




In this case $V_1 \cap V(H)$ and $V_2 \cap V(H)$ will be the partite sets of $H$ and so $H$ will be bipartite

<u>Case (ii)</u> $V(H) \cap V_1 = \emptyset$ or $V(H) \cap V_2 = \emptyset$
In this case $H$ will be a non-trivial graph with no edges. So it will be trivially bipartite.

1.25 Let $V_1 = \{u_1, \ldots, u_k\}$ and $V_2 = \{v_1, \ldots, v_\ell\}$ be the partite sets of $G$. If we arrange the vertices as shown below, then the adjacency matrix will have the required form

$$\begin{array}{cc} & \begin{matrix} u_1 \cdots u_k & v_1 \cdots v_\ell \end{matrix} \\ \begin{matrix} u_1 \\ \vdots \\ u_k \\ v_1 \\ \vdots \\ v_\ell \end{matrix} & \left[\begin{array}{c|c} \underset{\text{(zeros)}}{O} & A \\ \hline B & O \end{array}\right] \end{array}$$

because there are no edges between vertices in $V_1$ and there are no edges between vertices in $V_2$. Also if $u_i$ is adjacent to $v_j$, then $v_j$ is adjacent to $u_i$. So $B = A^T$.

1.27 Hint: First show that $p \cdot \delta(G) \leq$ sum of degrees $\leq p\Delta(G)$
Now use the fact that sum of degrees $= 2q$.

1.28 Hint: The only way to get a walk of length 2 from $v_i$ to $v_i$ is to go along an edge and come back along it.

1.29 Let $p$ be the no. of vertices in $G$. There are 2 cases
Case (i) : $G$ has a vertex of degree 0: In this case the possible degrees in $G$ are $0, 1, 2, \ldots, p-2$. So we have $p$ vertices and $p-1$ possible degrees. Hence 2 vertices must have the same degree.
Case (ii) : $G$ has no vertices of degree 0: In this case the possible degrees in $G$ are $1, 2, \ldots, p-1$. So we have $p$ vertices & $p-1$ possible degrees. So two vertices must have the same degree.

# CHAPTER 2

2.1 (a) The <u>graph distance</u> is the distance function obtained by assigning a weight of 1 to each edge in the graph. It is a metric function bec.

(i) First $d(x,x) = 0$ and if $x \neq y$, then $d(x,y)$ will be a positive integer. So $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,y)$ = length of shortest path from $x$ to $y$
= length " " " $y$ to $x$
= $d(y,x)$

So $d(x,y) = d(y,x)$.

(iii) If we take the shortest path from $x$ to $y$ and then the shortest path from $y$ to $z$ we will get a walk from $x$ to $z$. From this walk we can extract a path $P$ from $x$ to $z$. This path $P$ will then have length $\leq d(x,y) + d(y,z)$. Since $d(x,z)$ is the length of the shortest path from $x$ to $z$ we must have $d(x,z) \leq$ length of $P \leq d(x,y) + d(y,z)$. $\therefore$ $d(x,y) + d(y,z) \geq d(x,z)$.

(b) In a general weighted graph conditions (ii) and (iii) above, will still be true. However, if we have negative weights then we won't have $d(x,y) \geq 0$ for all $x, y$. And if we have zero weights, we won't have $d(x,y) = 0$ if and only $x = y$. If all weights are positive we will, however, get a metric function.

2.2. <u>Hint</u>: Start with all vertices adjacent to x and label each one with a 1 and the edge connecting it to x. Then go to the vertices which are adjacent to those with label 1. Label each of these vertices with a 2 and the two edges connecting it to x, and so on.

2.3. Stop the algorithm as soon as the vertex Y is label. This will get you the distance from x to a specified vertex y.

2.6. Remove the part about stopping when the vertex Y is deleted from T. Keep going until all vertices in G that are reachable from x has been deleted from T. Then stop. (The version that I gave in class gives the distance from x to all the vertices in G.)

2.8. <u>Hint</u>: Start at any vertex, $v_1$ say. Then there must be an edge $e_1$ from $v_1$ to some other vertex, $v_2$ say (otherwise $v_1$ would be an isolated vertex and the graph would be disconnected). Let $G_1 = \langle \{v_1, v_2\}, \{e_1\} \rangle$. Then we must have an edge $e_2$ from $G_1$ to some new vertex, $v_3$ say, in G. Let $G_2 = \langle \{v_1, v_2, v_3\}, \{e_1, e_2\} \rangle$. Then we can find an edge $e_3$ from $G_2$ to a new vertex, $v_4$ say, in G and so on. At the end $G_{p-1}$ will be a subgraph of G with p-1 edges. So G must have at least p-1 edges.

2.9 If $q > (p-1)(p-2)/2$, then this will ensure that any $(p,q)$ graph will be connected. We will prove this by show that any disconnected graph with $p$ vertices has at most $(p-1)(p-2)/2$ edges.

Let $G$ be a disconnected graph with $p$ vertices. Then we can split $G$ into two disjoint pieces $G_1$ (with $k$ vertices) and $G_2$ (with $p-k$ vertices) Here $k$ is an integer with $1 \le k \le p-1$.

$k$ vertices


$p-k$ vertices

Now $|E(G_1)| \le k(k-1)/2$ and $|E(G_2)| \le (p-k)(p-k-1)/2$ because a graph with $n$ vertices has at most $n(n-1)/2$ edges. So $G$ has at most

$$k(k-1)/2 + (p-k)(p-k-1)/2 \quad \text{edges.}$$

But
$$\frac{(p-1)(p-2)}{2} - \frac{k(k-1)}{2} - \frac{(p-k)(p-k-1)}{2}$$
$$= \frac{1}{2}\left[(p^2-3p+2) - (k^2-k) - (p^2-2kp-p+k^2+k)\right]$$
$$= \frac{1}{2}\left[p^2-3p+2-k^2+k-p^2+2kp+p-k^2-k\right]$$
$$= \frac{1}{2}\left[2kp-2p-2k^2+2\right] = kp-p-k^2+1$$
$$= \underbrace{(k-1)}_{\ge 0}\underbrace{(p-k-1)}_{\ge 0} \ge 0 \quad \leftarrow \text{because } 1 \le k \le p-1$$

Hence $|E(G)| = \frac{k(k-1)}{2} + \frac{(p-k)(p-k-1)}{2} \le \frac{(p-1)(p-2)}{2}$.

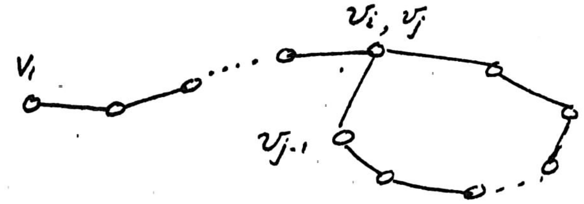
So if $q > (p-1)(p-2)/2$ the $(p,q)$ graph will be connected.

8 If $q = (p-1)(p-2)/2$, this is not sufficient to guarantee connectedness, because $K_1 \cup K_{p-1}$ has $p$ vertices and

2.11 Start at any vertex, $v_1$ say, in the circuit. Continue along the circuit until a vertex is repeated for the first time. We will then get a sequence

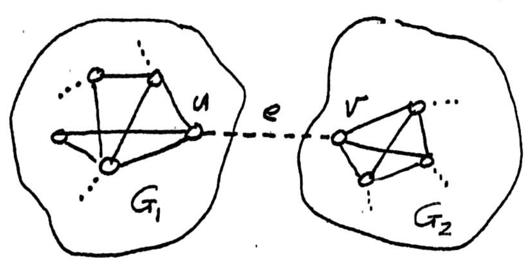
$$v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_{j-1}, v_j = v_i$$

in which all the vertices $v_1, \ldots, v_{j-1}$ are all distinct and $v_j = v_i$.



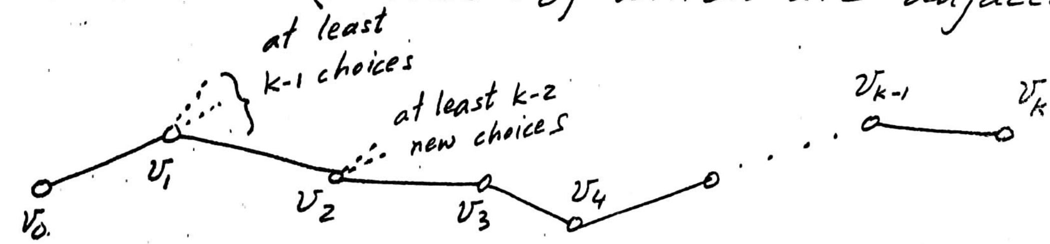

Since $v_i, v_{i+1}, \ldots, v_{j-1}$ are all distinct, it follows that $v_i, v_{i+1}, \ldots, v_{j-1}, v_j = v_i$ is a cycle that is contained in the circuit.

2.16 Suppose all the vertices in $G$ are of even degree and $G$ has a bridge $e = uv$. Then $G - \{e\}$ consists of two disjoint components $G_1$ and $G_2$




Now the only changes in the degrees were that the degrees of $u$ and $v$ were reduced by 1 in $G - \{e\}$. Since all the vertices in $G$ were originally of even degree it follows that the sum of the degrees in $G_1$ and the sum of the degrees in $G_2$ are both odd – which is impossible. Hence $G$ cannot have any bridges.

2.20 Let $k = \delta(G)$. Start at any vertex, $v_0$ say. Then go to another vertex, $v_1$ say, which is adjacent to $v_0$. Since $\deg(v_1) \geqslant k$, there are at least $k-1$ vertices (besides $v_0$) which are adjacent to $v_1$.




Choose one of these $k-1$ vertices, say it is $v_2$. Then look at $v_2$. Since $\deg(v_2) \geqslant k$, there are at least $k-2$ new vertices (besides $v_0$ and $v_1$) which are adjacent to $v_2$. Choose one of these $k-2$ new vertices, say it is $v_3$, and then look at $v_3$ . . . and so on. When we get to $v_{k-1}$ we will be guaranteed that there is at least 1 new vertex (besides $v_0, v_1, \ldots, v_{k-2}$) which is adjacent to $v_{k-1}$ (because $\deg(v_{k-1}) \geqslant k$). Now if we call this last new vertex $v_k$, we will get a path

$$v_0, v_1, \ldots, v_{k-1}, v_k$$

of length $k = \delta(G)$ as required.

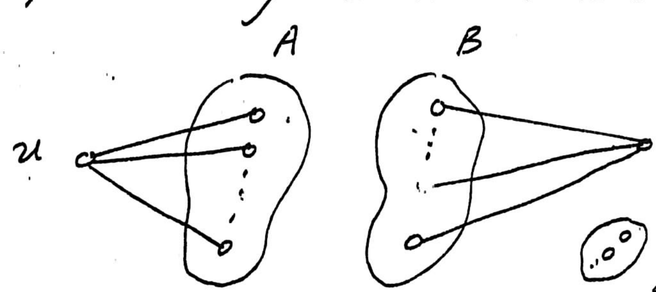
2.21 Let $P$ be the statement "$G$ is connected" and $Q$ be the statement "For every partition of $V(G)$ into two non-empty sets $V_1$ & $V_2$, there is an edge from a vertex in $V_1$ to a vertex in $V_2$"

To prove that $P \Leftrightarrow Q$ it will suffice to show that $P \Rightarrow Q$ and $\neg P \Rightarrow \neg Q$ (Remember $\neg P \Rightarrow \neg Q$ is logically equivalent to $Q \Rightarrow P$)

2.21. $P \Rightarrow Q$: Suppose $G$ is connected. Let $V_1$ & $V_2$ be a partition of $V(G)$. Then we can find a path from a vertex in $V_1$ to a vertex in $V_2$ (bec. $G$ is conn.) Go along this path until you reach a vertex $v$ in $V_2$ for the first time. Then the previous vertex, $u$ say, will be in $V_1$. So we get an edge $uv$ from $V_1$ to $V_2$. $\therefore$ $Q$ is true. So $P \Rightarrow Q$

$\neg P \Rightarrow \neg Q$: Suppose $G$ is not connected. Then we can split $G$ into two parts $G_1$ & $G_2$ such that $G_1$ and $G_2$ are disjoint. Let $V_1 = V(G_1)$ and $V_2 = V(G_2)$. Then there will no edge from $V_1$ to $V_2$. So $\neg Q$ will be true. $\therefore$ $\neg P \Rightarrow \neg Q$.

2.22 Take any two vertices $u$ and $v$ in $G$. Suppose there is no path from $u$ to $v$. Let $A$ = set of vertices adjacent to $u$, and $B$ = set of vertices adj. to $v$. Then $A$ and $B$ are disjoint (otherwise we would have a path of length 2 from $u$ to $v$).



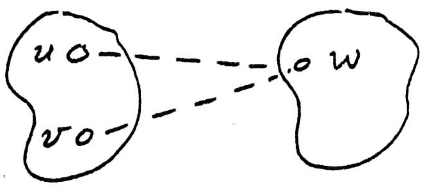

← other vertices which are adjacent to neither $u$ nor $v$.

Also, since $G$ has $p$ vertices and $u$ and $v$ are not in $A$ or $B$, we must have $|A| + |B| \leq p-2$. So

$$|A| \leq (p-2)/2 \quad \text{or} \quad |B| \leq (p-2)/2.$$

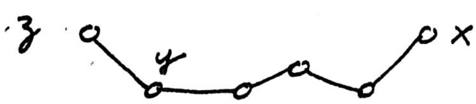
Hence $\deg(u) = |A| \leq (p-2)/2$ or $\deg(v) = |B| \leq (p-2)/2$.
But this contradicts the fact that $\delta(G) \geq (p-1)/2 > (p-2)/2$.
So there must be a path from $u$ to $v$. Hence $G$ is connected

2.24 Let $u$ and $v$ be any two vertices in $G$. If $uv \notin G$ then $uv \in \bar{G}$, so we get a path from $u$ to $v$ in $\bar{G}$. And if $uv \in G$, then $u$ and $v$ are in the same component of $G$. Choose a vertex $w$ in another component of $G$.



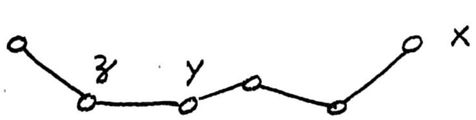

Then $uw$ and $wv$ are edges in $\bar{G}$. So we again get a path (namely $u, w, v$) from $u$ to $v$ in $\bar{G}$. Hence $\bar{G}$ is connected

2.25 Suppose $G$ is not complete. Then we can find two vertices $x$ and $z$ which are not adjacent. Now look at a path from $z$ to $x$. Let $y$ be the vertex immediately after $z$.




If $y$ is adjacent to $x$, then we will be done (because we will have $xy$ & $yz \in E(G)$ but $xz \notin E(G)$).

Now if $y$ is not adjacent to $x$, then we shift the $z$ and $y$ one step towards $x$ as shown below




If this new $y$ is adjacent to $x$, then we will be done. Otherwise we keep shifting the $z$ and $y$ one step towards $x$ until $y$ is adjacent to $x$ for the first time. This will then give us $xy$ & $yz \in E(G)$ and $xz \notin E(G)$ as required

2.29 Make a graph with triples $\langle a,b,c\rangle$ as vertices, where

$a$ = no. of missionaries on far bank

$b$ = no. " cannibals " " "

$c$ = 1 (if boat is on far bank), 0 (if not)

We need a path from $\langle 0,0,0\rangle$ to $\langle 3,3,1\rangle$. Proceed as follows:

$\langle 0,0,0\rangle$

← two cann. take boat to far side

$\langle 0,2,1\rangle$

← one cann. returns to near bank with boat

$\langle 0,1,0\rangle$

← two remaining cann. take boat to far side

$\langle 0,3,1\rangle$

⋮

$\langle 3,3,1\rangle$ goal reached

2.30 It will suffice to show that there is no path of length $\leq 11$ from $\langle 0,0,0\rangle$ to $\langle 3,3,1\rangle$ in the graph obtained in 2.29

2.31 A four cannibals & 4 missionaries problem does not make sense because there is no way (with the notation of 2.29) of getting from $\langle 0,0,0\rangle$ to $\langle 4,4,1\rangle$

2.32 Make a graph with 4-tuples $\langle a,b,c,d\rangle$ as vertices, where

$d$ = 1 (if car is in town), 0 (if not)

$a, b, c$ = weights of 1st, 2nd & 3rd couples resp. in town (husband = 1, wife = 2)

We need a path from $\langle 0,0,0,0\rangle$ to $\langle 3,3,3,1\rangle$. Proceed thus:

$\langle 0,0,0,0\rangle$

← 1st & 2nd wives go to town with car

$\langle 2,2,0,1\rangle$

← 1st wife returns with car

$\langle 0,2,0,0\rangle$

← 1st & 3rd wife go to town

$\langle 2,2,2,1\rangle$

← 3rd wife returns with car

$\langle 2,2,0,0\rangle$

← 1st & 2nd husbands go to town

$\langle 3,3,0,1\rangle$

⋮

$\langle 3,3,3,1\rangle$ goal reached.