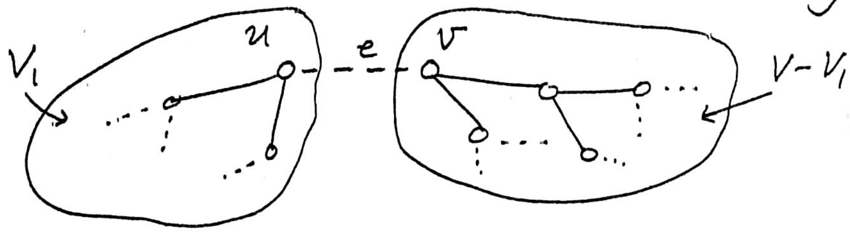
CHAPTER 3

3.1 We have a tree $T = \langle V, E \rangle$ and an edge $e \in E$. We want to show that $T-e$ consists of exactly two components. Let u & v be the endpoints of e. Put $V_1 = \{x \in V :$ there is a path from x to u which does not use the edge $e\}$. Then the vertices of V_1 will all be mutually accessible from each other. (So the part of $T-e$ that is induced by V_1 will be connected.) Now look at $V - V_1$. Each vertex y in $V-V_1$ can be reached from u only by using the edge e. So if we start at v we will be able to reach all the vertices of $V-V_1$ without using the edge e.

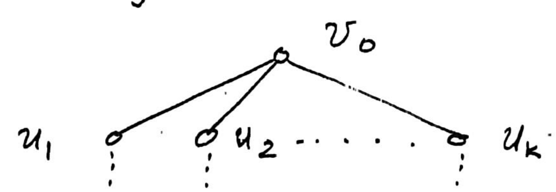


Thus the vertices in $V-V_1$ will all be mutually accessible. Since there is only one path in T from u to v, there is no path in $T-e$ from u to v. Hence no vertex in V_1 will be accessible from a vertex in $V-V_1$ in $T-e$. So we will have exactly two components.

3.2. This is the same problem as 2.8 We will give a slightly different proof here. First we remove edges that are in a cycle of G (one at a time) until we get a graph

3.2 with no cycles. Since G was connected to begin with, this new graph will be connected also. This will give us a spanning tree T of G. Now we know that a tree with p vertices has $p-1$ edges. So T has $p-1$ edges. Hence G must have had at least $p-1$ edges to start with.

3.3. First of all there is a mispint in the problem. The problem should read: " Show that if $\Delta(T)=k$, then T has at least k leaves."
Let v_0 be a vertex with $\deg(v_0) = k$. If we designate v_0 as the root, then we will get a rooted tree $\langle T, v_0 \rangle$. Now let $u_1, \dots, u_k$ be the vertices adjacent to v_0.



Then u_1 is a leaf or one of its descendants is a leaf. The same thing is true for u_2, $\dots, u_k$. So we are assured of getting at least k leaves. (Note: u_1 and u_2 cannot be the ancestor of the same leaf – otherwise we will get a cycle in the tree).

3.5. We will show that $d_1, \dots, d_p$ is the degree sequence of a non-trivial tree $\Leftrightarrow$ $d_1, \dots, d_p$ are ≥ 1 and $d_1 + d_2 + \dots + d_p = 2(p-1)$. (This is slightly different from what the textbook is asking you to prove.)

3.5 ($\Rightarrow$) Suppose $d_1, \ldots, d_p$ is the degree seq. of a non-trivial tree T. Then $d_1, \ldots, d_p$ are all ≥ 1 because each vertex is of degree ≥ 1. (If we had a vertex of deg 0, it will contradict the fact that we have a non-trivial tree). Also

$$\begin{aligned} d_1 + d_2 + \cdots + d_p &= \text{sum of degrees in } T \\ &= 2\,(\text{no. of edges in } T) \\ &= 2(p-1). \end{aligned}$$

($\Leftarrow$) Suppose $d_1, \ldots, d_p$ are all ≥ 1 and $d_1 + \cdots + d_p = 2(p-1)$. We have to find a tree T with degree sequence $d_1, \ldots, d_p$. We will prove this by induction on p. If $p = 2$, then $d_1 + d_2 = 2(2-1) = 2$. Since $d_1, d_2 \geq 0$, we must have $d_1 = d_2 = 1$. The tree $K_2 =$ o—o has d_1, d_2 as its deg. seq., so the result is true for $p=2$.

Now suppose the result is true for $p \geq 2$. Let $d_1, \ldots, d_{p+1}$ be any seq. of positive integers with $d_1 + d_2 + \cdots + d_{p+1} = 2(p+1-1)$. Then one of the d_i's must be 1 and another of the d_i's must be greater than 1 (because if all the d_i's were 1's we will get $d_1 + d_2 + \cdots + d_{p+1} = p+1 \neq 2(p+1-1)$; and if all the d_i's were ≥ 2, we will get $d_1 + d_2 + \cdots + d_{p+1} \geq 2(p+1) > 2(p+1-1)$.) Let us say that $d_1 > 1$ and $d_{p+1} = 1$ (to keep notation simple). Then the new sequence

$$d_1 - 1,\ d_2,\ \ldots,\ d_p$$

satisfies the conditions that all terms are ≥ 1 and

$$(d_1 - 1) + d_2 + \cdots + d_p = 2(p-1)$$

3.5 So by the induction hypothesis we can find a tree T_p with $d_1-1, d_2, \ldots, d_p$ as its degree sequence. Now if we add an edge from the vertex with degree d_1-1 to a new vertex, we will get a tree T_{p+1} with degree sequence $d_1, d_2, \ldots, d_p, d_{p+1}=1$. So the result is true for $p+1$. Hence by the Principle of Math. Ind., the result is true for all p.

3.6 Let $k =$ largest degree in $T = \Delta(T)$ and for each $i = 1, \ldots, n$ let

$$n_i = \text{no. of vertices of degree } i.$$

Then $\quad n_1 + n_2 + \cdots + n_k =$ no. of vertices in $T = p$

and $\quad 1.n_1 + 2.n_2 + \cdots + k.n_k =$ sum of degrees in T

$$= 2.|E(T)| = 2(p-1).$$

So

$$1.n_1 + 2.n_2 + 3.n_3 + \cdots + k.n_k = 2p-2 \qquad (1)$$

$$2.n_1 + 2.n_2 + 2.n_3 + \cdots + 2.n_k = 2p \qquad (2)$$

$$\therefore \quad -n_1 + 0 + (3-2)n_3 + \cdots + (k-2)n_k = -2 \ : \ (1)-(2)$$

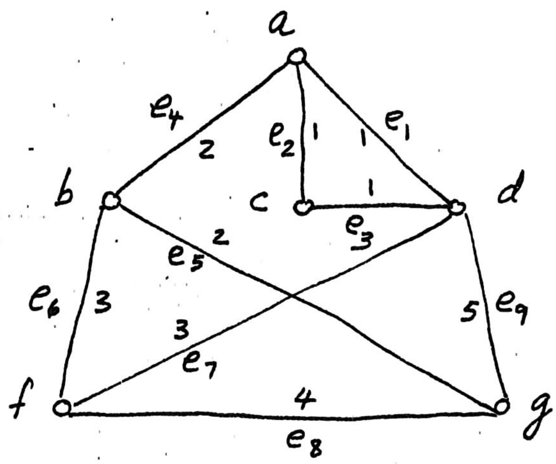
So

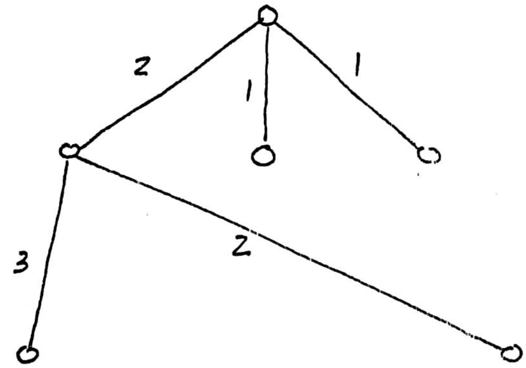
$$n_1 = 2 + (3-2)n_3 + \cdots + (k-2).n_k$$

$$= 2 + \sum_{i=3}^{k} (i-2)n_i = 2 + \sum_{\deg(v)\geq 3} \deg(v) - 2$$

$\therefore$ no. of leaves in $T = 2 + \sum_{\deg(v)\geq 3} \deg(v) - 2$

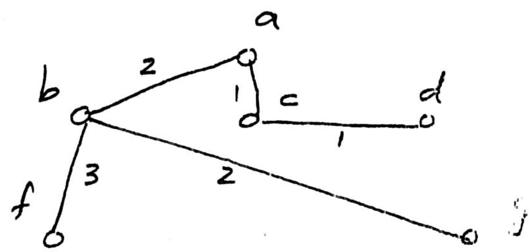
3.8.





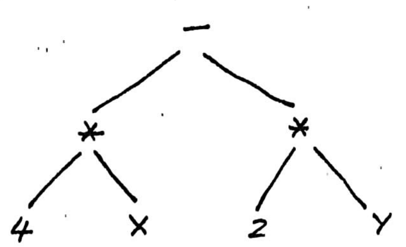
e_1, e_2, e_4, e_5, e_6 : $\quad w(T) = 9$

3.9 Let us start with the vertex f. Then
1st edge is fb (could have chosen fd)
2nd " is ba (could have chosen bg)
3rd " is ac
4th " is cd (could have chosen ad)
5th " is bg
So we get:

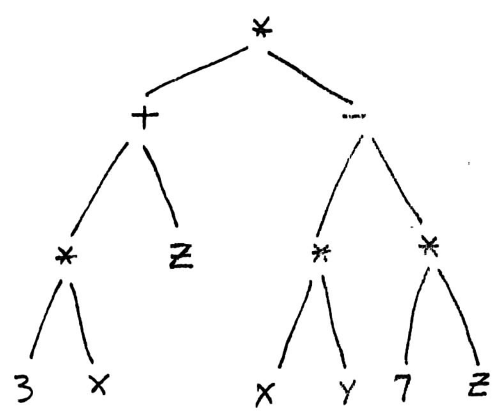


$w(T) = 9$.

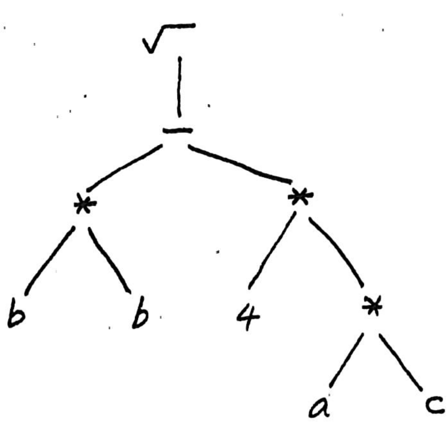
3.12. (a)



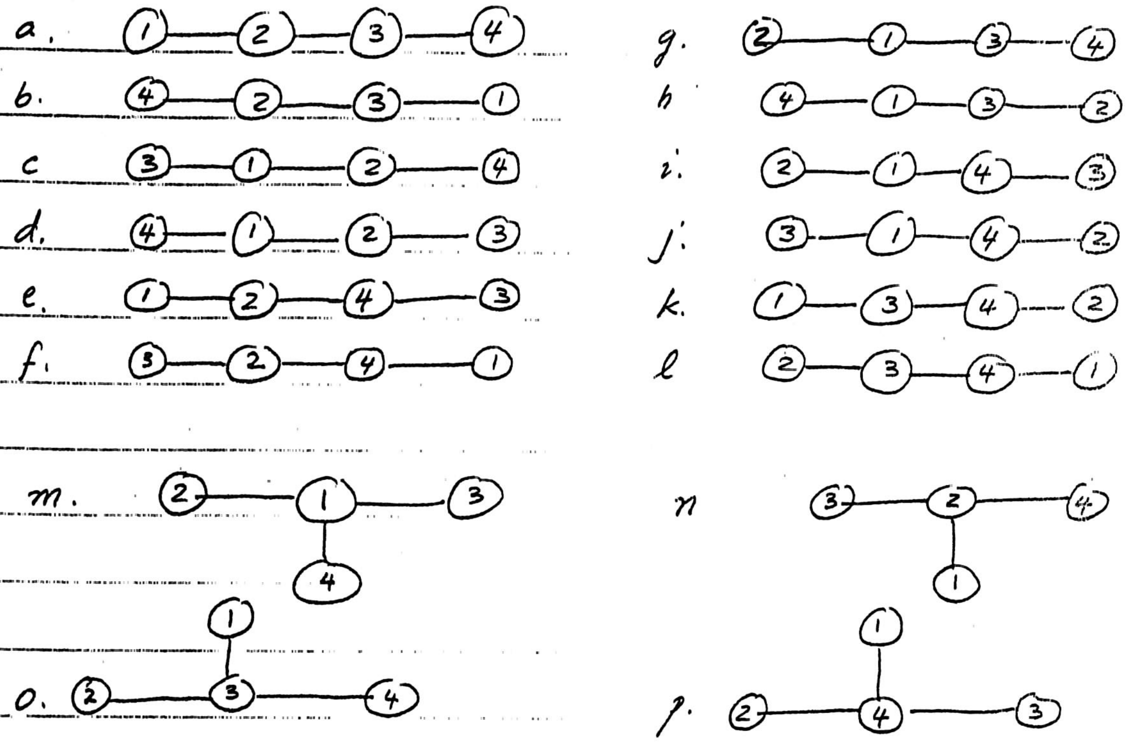
(b)



(c)



3.15



3.16 The sequences corresponding to these trees are as follows:

a. ⟨2,3⟩	e ⟨2,4⟩	i. ⟨1,4⟩	m. ⟨1,1⟩
b, ⟨3,2⟩	f. ⟨4,2⟩	j. ⟨4,1⟩	n, ⟨2,2⟩
c, ⟨1,2⟩	g. ⟨1,3⟩	k. ⟨3,4⟩	o. ⟨3,3⟩
d, ⟨2,1⟩	h. ⟨3,1⟩	l. ⟨4,3⟩	p. ⟨4,4⟩

3.24 X, Y, Z, a, b, c

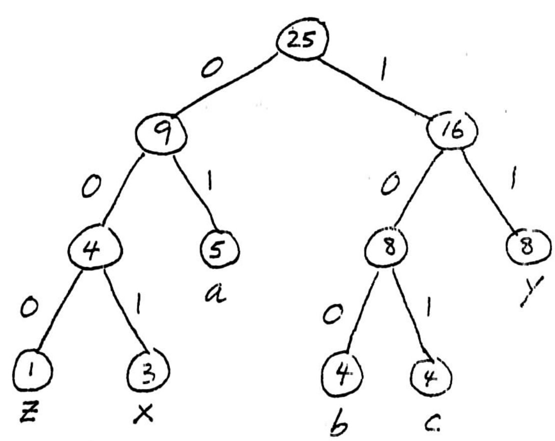
~~3~~, 8, ~~1~~, 5, 4, 4

8, 5, 4, ~~4~~, ~~4~~

8, ~~5~~, ~~4~~, 8

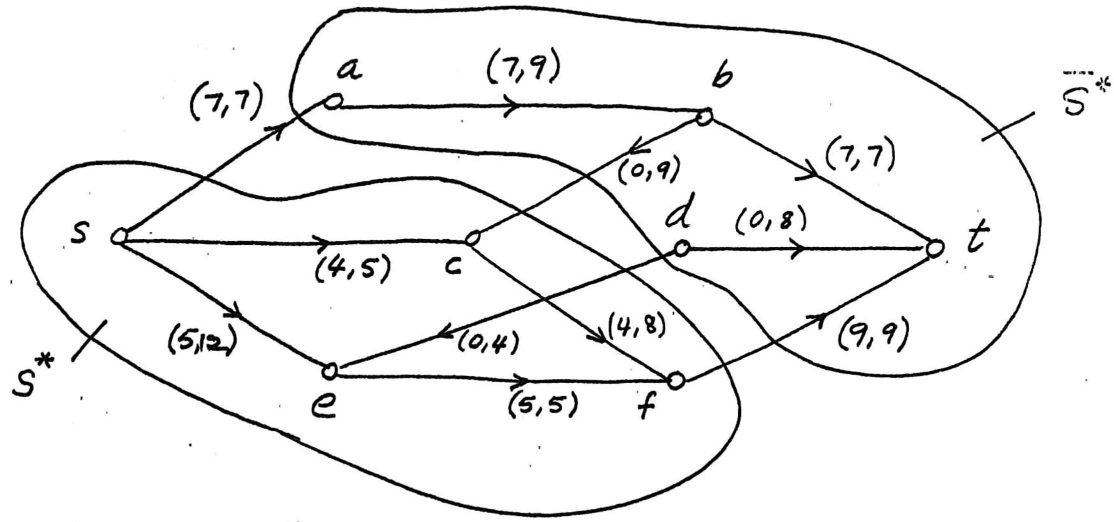
~~8~~, ~~8~~, 9

9, 16



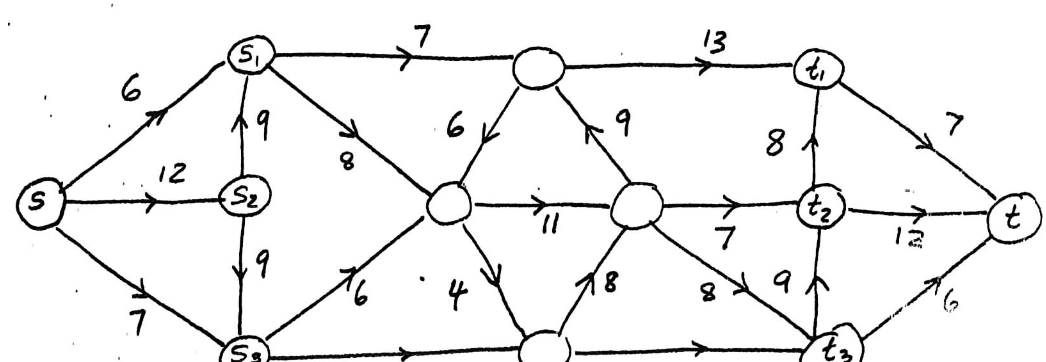
Char.	X	Y	Z	a	b	c
Code	001	11	000	01	100	101

4.1 A maximal flow f^* is shown in the network below. The value of this maximal flow is given by $F(f^*) = 16$. (There are other maximal flows but they will all have the same value.)



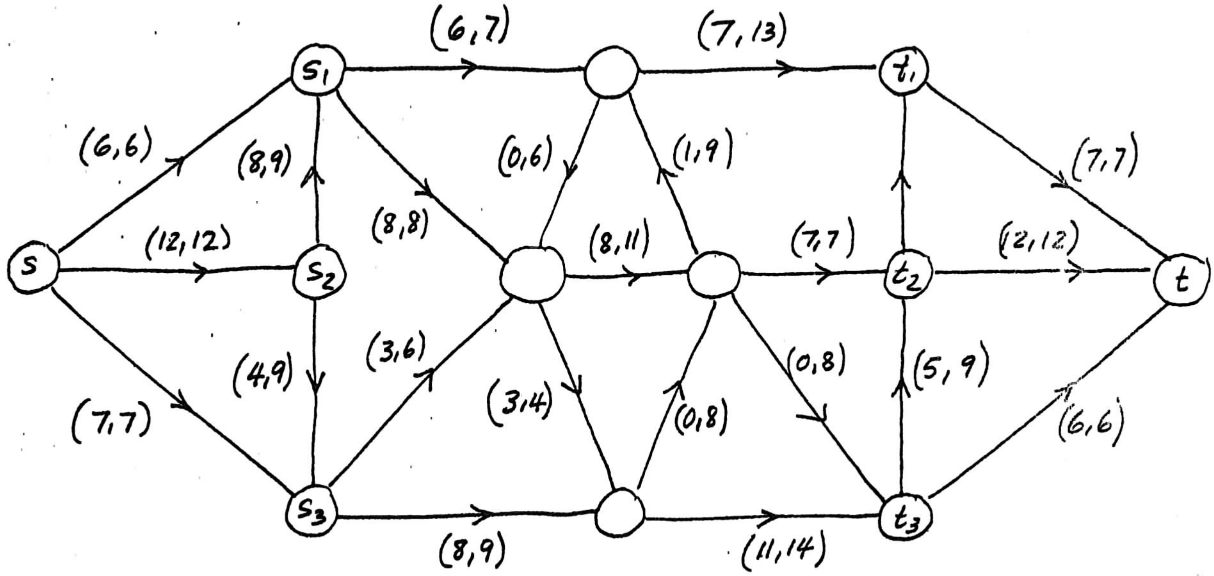
Min Cut $S^* = \{s, c, e, f\}$. Notice all the edges from S^* to $\bar{S}^*$ are full and all the edges from $\bar{S}^*$ to S^* are empty of flow.

4.5 Add a new source s and a new sink t with edges having capacities as shown below. Then find a maximal flow in the new network

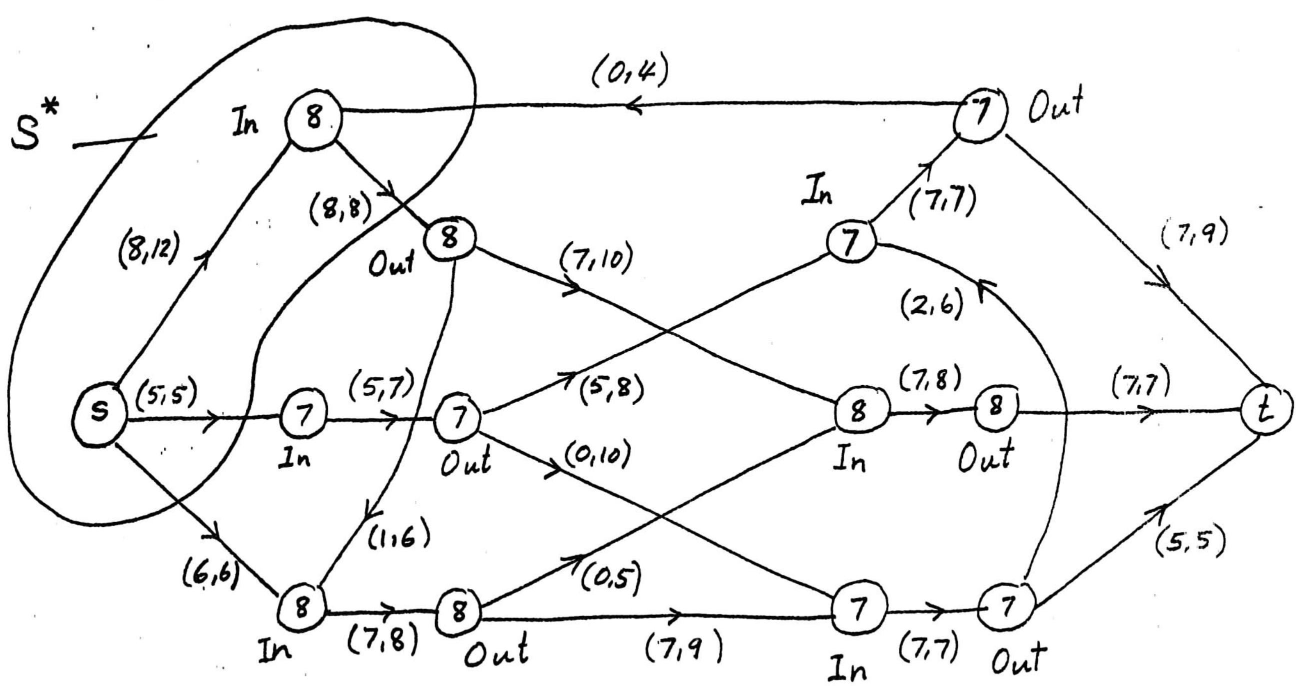


Ans: The maximal flow will show that the demands can be met simultaneously.

4.5



4.9 Replace each of the vertices except s and t by two copies, an "IN" and an "OUT", and connect them as shown below. Then find a maximal flow f^* in the new network. The value of this maximal flow is given by $F(f^*) = 19$.



(21) Min Cut $S^* = \{s, ⑧ In\}$