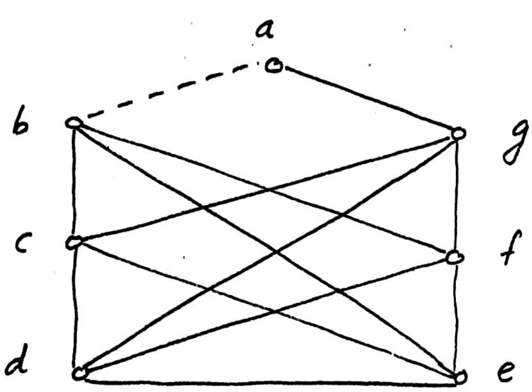
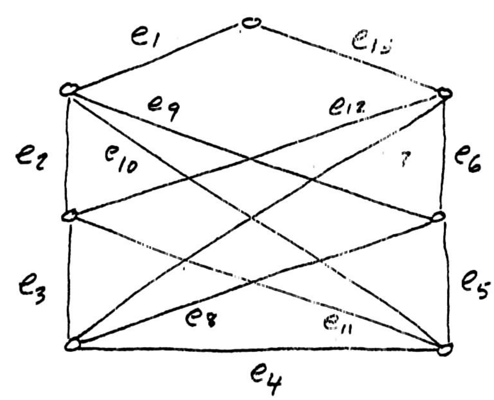
CHAPTER 5

5.1. Partition the $2k$ vertices into k pairs $\{u_1, v_1\}$, $\{u_2, v_2\}$, - - - , $\{u_k, v_k\}$. Then add an edge between u_i and v_i for each $i = 1, \ldots, k$. This will give us a connected multi-graph G' in which all vertices are of even degree. By Theoreom 5.1.1, it follows that G' has an Euler circuit Q. Now if we delete the edges that we added we will get k open trails in G.

5.4 We will find an Euler circuit of the graph by using Fleury's algorithm. First label the vertices as shown

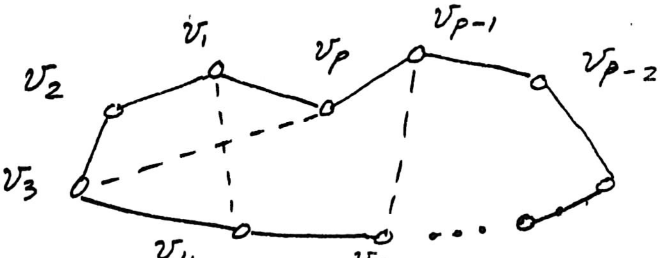




$e_1, e_2, e_3, \ldots, e_{13}$ is one Euler circuit that can be obtained from Fleury's algorithm. Since this is a graph, we can also use the vertex sequence $a, b, c, d, e, f, g, d, f, b, e, c, g, a$ to represent the same Euler circuit.

Note: There are many other Euler circuits of this graph.

5.10 Suppose G is Hamiltonian (this just means that G has a Hamilton cycle). Then we can find a Hamilton cycle with vertex seq., say $v_1, v_2, v_3, \ldots, v_p, v_1$



The broken lines represent edges in G that are not in the Hamilton cycle.

Now if S is any proper subset of $V(G)$ then we can write $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ for some $k < p$ and $1 \le i_1 < i_2 < \ldots < i_k \le p$. Now if we remove v_{i_1}, the graph will still be connected (because it contained a Ham. cycle to start with). And if we remove v_{i_2} it will have ≤ 2 components (because all the vertices of $V(G) - \{v_{i_1}\}$ were joined by a path — deleting v_{i_2} can split this path but it might not always leave a disconnected graph). Similary, if we remove v_{i_3}, we will get ≤ 3 components, ... and so on. When we remove v_{i_k} we will be assured of having $\le k$ components. Thus $G - S$ will have at most $k = |S|$ components.

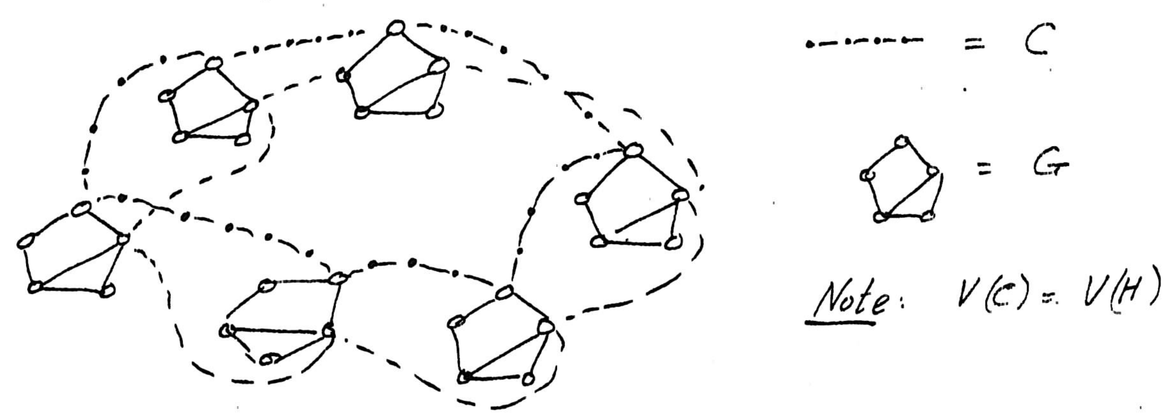
5.11 Suppose G is Hamiltonian. Then we can find a Hamilton cycle in G. So if we take out

5.11 any vertex from G, the remaining graph will still be connected and have at least 2 vertices (Remember we need at least 3 vertices to have a cycle in a graph). So we must take out at least 2 vertices in order to disconnect G or reduce it to K_1. Hence $k_v(G) \geq 2$, i.e., G must be 2-connected.

So if G is Hamiltonian, then G is 2-connected. Hence if G is not 2-connected, then G cannot be Hamiltonian.

5.13 If G and H are Hamiltonian, then $G \times H$ is Hamiltonian.

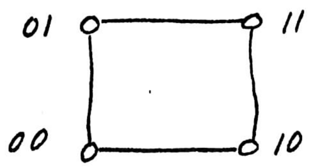
<u>HINT:</u> Let C be a Hamilton cycle in H. Then $G \times C$ is a subgraph of $G \times H$. Now $G \times C$ is essentially many identical copies of G strung out in a cycle.



So if we take a Hamilton path (= a Hamilton cycle − one edge) in each copy of G and then jump to another copy by using the edges of C. we will be able to get a Hamilton cycle of $G \times C$. Since $G \times C$ is a subgraph of $G \times H$, we will then have a Hamilton cycle of $G \times H$

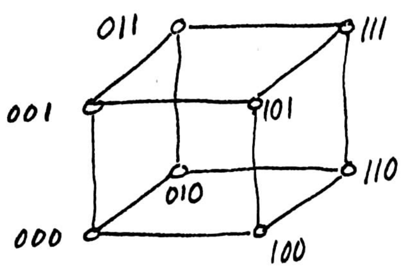
5.15 Label the vertices of the 2-, 3-, and 4-cubes by the seq. of 0's and 1's that represent their coordinates.

(a)



Ham. cycle: 00, 10, 11, 01, 00

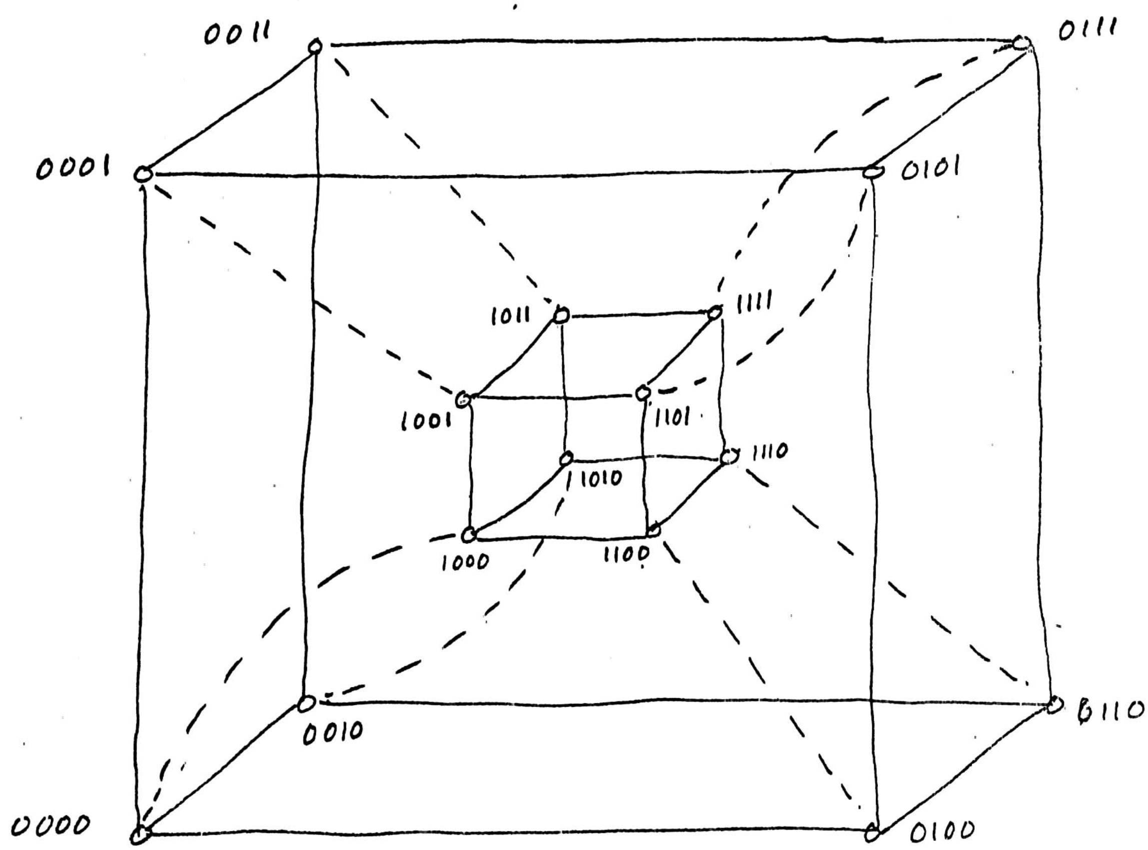
(b)



Ham. cycle:

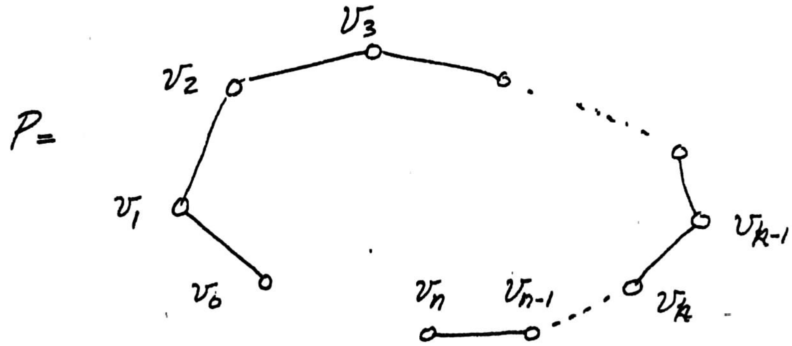
000, 001, 011, 010, 110, 111, 101, 100, 000

(c)



Ham. cycle: 1000, 1001, 1011, 1010, 1110, 1111, 1101, 1100, 0100, 0110, 0010, 0011, 0111, 0101, 0001, 0000, 1000

5.16. Suppose G is a graph with $\delta(G) \geq 2$. Let $k = \delta(G)$. We have to show that G contains a cycle of length at least $k+1$. Now we know from problem 2.20 that G contains a path of length at least k. Let P be the longest possible path in G. Then the length of P must be $\geq k$. So if $v_0, v_1, v_2, \ldots, v_n$ is the vertex sequence of P, then $n \geq k$.



Now consider our longest path P. If v_n is adjacent to v_0, then we will get a cycle

$$v_0, v_1, v_2, \ldots, v_n, v_0$$

of length $n+1 \geq k+1$ and we will be done.

So suppose v_0 is not adjacent to v_n. Then v_0 must be adjacent to v_i for some $i \geq k$ because $\deg(v_0) \geq \delta(G) = k$. (This is because v_0 cannot be adjacent to some new vertex that is different from $v_1, \ldots, v_n$ – otherwise this will give us a path that is longer than P. And if v_0 is adjacent only to $v_1, \ldots, v_{k-1}$ it will have degree $< k$). So we will get a cycle

$$v_0, v_1, v_2, \ldots, v_k, \ldots, v_{i-1}, v_i, v_0$$

of length $i+1 \geq k+1$. Hence in either case we get a cycle of length $\geq k+1$.

5.17. We will show that if G is not Hamiltonian, then $q < (p^2-3p+6)/2$. From this it follows that if $q \geq (p^2-3p+6)/2$, then G is Hamiltonian.

Suppose G is not Hamiltonian. Now if all vertices have degree $\geq p/2$, then G will be Hamiltonian by Corollary 5.2.1. So G must have a vertex, v_1 say, with $\deg(v_1) < p/2$. Let $k = \deg(v_1)$. We claim that G must also have a vertex v_2 with $\deg(v_2) \leq (p-1)-k$. There are 2 two cases:

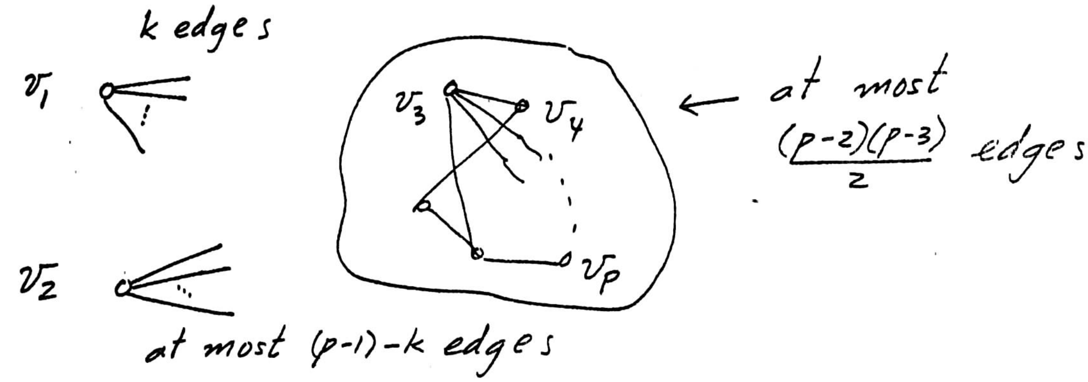
<u>Case (i)</u>: v_1 is the only vertex of degree $< p/2$. In this case we note that if all the other vertices were of degree $\geq p-k$, then G would have a Hamilton cycle by theorem 5.2.1. So G must a vertex v_2 with $\deg(v_2) < p-k$. Thus we will have $\deg(v_2) \leq (p-1)-k$.

<u>Case (ii)</u>: v_1 is <u>not</u> the only vertex of deg. $< p/2$. In this case we must have another vertex v_2 of degree $< p/2$. So $\deg(v_1) = k \leq (p-1)/2$ and $\deg(v_2) \leq (p-1)/2$. Thus

$$\deg(v_2) \leq (p-1)-(p-1)/2 \leq (p-1)-k$$

So in either case we get a vertex v_2 with $\deg(v_2) \leq (p-1)-k$. Now let us look at the maximum possible number of edges in G. Consider v_1 and v_2 as one part and $v_3, \ldots, v_p$ as the other.

5.17



v_1 has k edges incidented to it, v_2 has $\leq (p-1)-k$ edges incidented to it, and there can be at most $(p-2)(p-3)/2$ edges among $\{v_3, \ldots, v_p\}$. So

$$
\begin{aligned}
q &\leq (p-2)(p-3)/2 + k + (p-1) - k \\
&\leq (p-2)(p-3)/2 + p-1 \\
&< \frac{p^2-5p+6}{2} + p \\
&= \frac{p^2-5p+6}{2} + \frac{2p}{2} = \frac{p^2-3p+6}{2}.
\end{aligned}
$$

Thus $q < (p^2-3p+6)/2$ and we are done.

5.18 The proof is very similar to that of 5.17. Use Corollary 5.3.2 and Theorem 5.3.1, esp. in the two cases that arise.

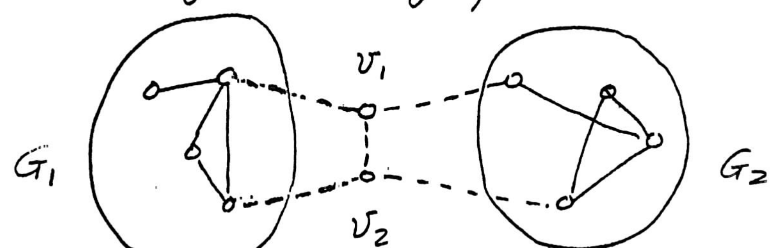
Note that

$$
\begin{aligned}
\binom{p-1}{2} + 3 &= \frac{(p-1)(p-2)}{2} + \frac{6}{2} \\
&= \frac{p^2-3p+2}{2} + \frac{6}{2} \\
&= \frac{p^2-3p+8}{2} = \frac{p^2-3p+6}{2} + 1
\end{aligned}
$$

So the two problems are indeed almost identical.

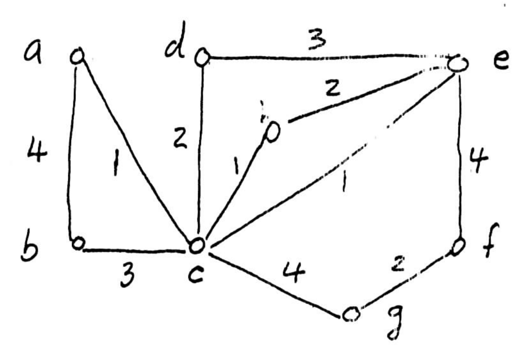
5.19 The problem is slightly misstated. It should read: "Show that if G is Ham.-connected and $p \geq 4$, then G is 3-connected." (compare with 5.11)

Suppose G is not 3-connected. Then $k_V(G) \leq 2$. Since G has at least 4 vertices, we cannot reduce it to K_1 by deleting 2 vertices. But we can certainly disconnect G by removing 2 vertices (bec. $k_V(G) \leq 2$). So let's take out two vertices, v_1 and v_2 say, and split the graph into two disjoint subgraphs G_1 and G_2.



Then there cannot be a Hamilton path from v_1 to v_2 in G (because if we go from v_1 to G_1, we will have to pass through v_2 before we get to G_2; and the same problem arises if we go to G_2 first.)
So G cannot be Hamilton-connected. Hence if G is not 3-connected then G is not Ham.-conn.
So if G is Ham.-conn., then G must be 3-connected

5.41 a, b, c, d, e, f, g, c, h, c, a is a minimum salesman walk. The total length of this walk is 25.
Another min. salesman walk is
a, b, a, c, d, c, h, c, e, f, g, c, a



CHAPTER 6

6.1 Let $G_1, \ldots, G_k$ be the k components of G and r_i be the number of regions into which the plane is divided by G_i. Then by Euler's Formula

$$r_i = q_i + 2 - p_i .$$

So
$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} (q_i + 2 - p_i)$$
$$= \sum_{i=1}^{k} q_i + \sum_{i=1}^{k} 2 - \sum_{i=1}^{k} p_i$$
$$= q + 2k - p$$

But in $\sum_{i=1}^{k} r_i$, we counted the infinite region k times (instead of once). So
$$r = \left(\sum_{i=1}^{k} r_i\right) - (k-1) = q + 2k - p - (k-1)$$
$$= q + k + 1 - p .$$

Thus $\quad p - q + r = 1 + k$.

6.5 We have a maximal planar graph G in which $n = \Delta(G)$ and p_i = no. of vertices of degree i $(i = 1, 2, \ldots, n)$. We also know that G has at least 4 vertices, so each vertex of G must be of degree ≥ 3 (because G is maximal planar). Thus $p_1 = p_2 = 0$. Now since G is maximal planar it has $3p - 6$ edges. Also

sum of degrees of G = 2 (no. of edges).

$$\therefore \quad 3 \cdot p_3 + 4 \cdot p_4 + 5 p_5 + \cdots + n \cdot p_n = 2 \cdot (3p - 6)$$

$$\therefore \quad \sum_{i=3}^{n} i \cdot p_i = 6p - 12$$

$$\therefore \quad \sum_{i=3}^{n} i \cdot p_i = \left(6 \cdot \sum_{i=3}^{n} p_i\right) - 12 \qquad \text{bec. } \sum_{i=1}^{n} p_i = p$$

So
$$\sum_{i=3}^{n} (i - 6) \cdot p_i = -12 .$$

6.5 Thus $(3-6)p_3 + (4-6)p_4 + (5-6)p_5 + \cdots + (n-6)p_n = -12$

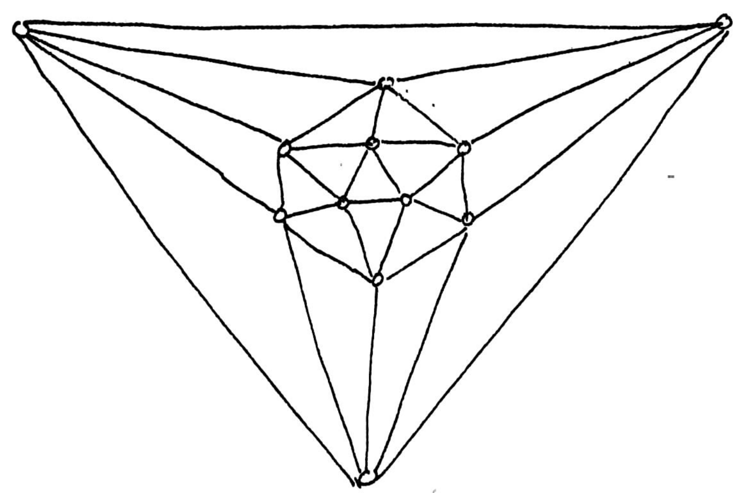
So

$$(7-6)p_7 + (8-6)p_8 + \cdots + (n-6)p_n + 12$$
$$= -(3-6)p_3 - (4-6)p_4 - (5-6)p_5 - (6-6)p_6$$

$$\therefore \quad p_7 + 2p_8 + 3p_9 + \cdots + (n-6)p_n + 12 = 3p_3 + 2p_4 + p_5$$

as required.

6.7 The icosahedral graph shown below is a planar graph in which no vertex is of degree less than 5.



Now that you have seen one example, find another.

6. Method I: First we add edges to G until we get a maximal planar graph G'. Since G had at least 4 vertices, all the vertices of G' will have degree $\geqslant 3$. We will show that G' has four vertices with degrees $\leqslant 5$. Since G' was formed by adding edges to G, these four vertices will also have degrees $\leqslant 5$ in G.

Now suppose G' did not have four vertices with degrees $\leqslant 5$. Then G' has at most three

6.8 vertices of degree ≤ 5. So we will have

$$\text{sum of degrees of } G' \geq 3.\underset{\substack{\uparrow \\ A}}{3} + (p-3).\underset{\substack{\uparrow \\ B}}{6}$$

(A represents the smallest possible degree of the three vertices of degree ≤ 5, and B represent the smallest possible degree of the remaining $p-3$ vertices)

So

$$\text{sum of degrees of } G' \geq 9 + 6p - 18 = 6p - 9$$

But G' has $3p-6$ edges (bec. it's maximal planar).

So $\quad 2\cdot(3p-6) = \text{sum of degrees of } G'$

$$\geq 6p - 9$$

$\therefore \quad 6p - 12 \geq 6p - 9$ — a contradiction.

Hence G' must have at least four vertices of degree ≤ 5.

<u>Method II</u>: We first add edges to get a maximal planar graph G' as above. Now from prob. 6.5

$$3p_3 + 2p_4 + p_5 = p_7 + 2.p_8 + \cdots + (n-6).p_n + 12$$

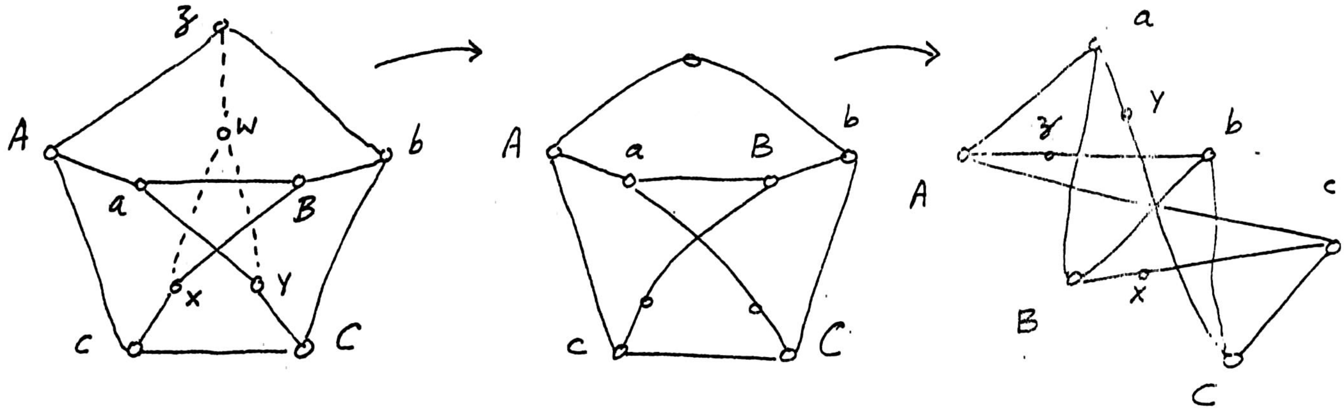
$$\therefore \quad 3p_3 + 2p_4 + p_5 \geq 12.$$

From this equation it follows that

$$p_3 + p_4 + p_5 \geq 4.$$

So G' must have at least four vertices which are of degree 3, 4 or 5. Thus our G will have at least four vertices of degree ≤ 5.

6.9 Let G be the subgraph of the Petersen graph obtained by deleting w and the edges adjacent to w. Then G is homeomorphic to $K_{3,3}$ and so, by Kuratowski's Theorem, the Petersen graph is non-planar.



6.10 Consider a planar embedding of G. Since all the cycles in G have length $\geq k$, each region in the planar embedding must be bounded by at least k edges. Since each edge can bound at most 2 regions we get

$$s_1 + s_2 + \cdots + s_r \leq 2.|E|$$

Here s_i is the number of edges on the boundary of the i-th region. Since $k \leq$ each s_i, we have

$$\underbrace{k + k + \cdots + k}_{r \text{ times}} \leq 2.|E|$$

$$\therefore \quad k.r \leq 2|E|$$

But $r = |E| + 2 - p$ by Euler's Formula

$$\therefore \quad k.(|E| + 2 - p) \leq 2|E|$$

So $$k.|E| + 2k - pk \leq 2.|E|$$

$$\therefore \quad (k-2)|E| \leq k(p-2)$$

and thus $|E| \leq k.(p-2)/(k-2)$.

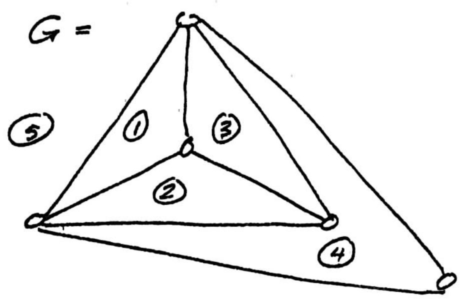
6.12 (a) Suppose G is self-dual. Then $G \cong G^*$.
But $V(G^*)$ = set of regions of G. Hence $|V(G^*)| = r$.
And since $|V(G^*)| = |V(G)|$ (because $G \cong G^*$) we get $|V| = |V(G)| = r$. Now

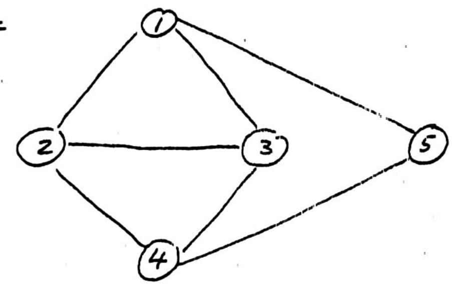
$$r = |E| - |V| + 2 \quad \text{by Euler's Formula}$$

So $|V| = |E| - |V| + 2$

$\therefore \quad 2|V| = |E| + 2$ as required.

(b)

$G =$ 

$G^* =$ 

In G, $|V| = 5$ and $|E| = 8$. So $2|V| = |E| + 2$.
But $G \not\cong G^*$ because G has 2 vertices of deg. 4 while G^* no vertices of deg. 4. So if $2|V| = |E| + 2$, it does not follow that G must be self-dual.

6.14 Let $p = |V(G)|$. Now suppose $p = 11$. Then

$$|E(G)| + |E(\bar{G})| = p(p-1)/2 = 55.$$

So $|E(G)| \geq 27$ or $|E(\bar{G})| \geq 27$

$$|E(G)| > 3.11 - 6 \quad \text{or} \quad |E(\bar{G})| > 3.11 - 6$$

So by Corollary 6.1.1, G is non-planar or $\bar{G}$ is non-planar. (Remember: if $q > 3p-6$, G is non-planar)

Now suppose $p \geq 12$. Then $|E(G)| + |E(\bar{G})| = p(p-1)/2$.

6.14 So $|E(G)| \geq p(p-1)/4$ or $|E(\bar{G})| \geq p(p-1)/4$.
But if $p \geq 12$, then

$$p(p-1)/4 \geq 12.(p-1)/4$$
$$\geq 3(p-1) = 3p-3 > 3p-6.$$

So $|E(G)| > 3p-6$ or $|E(\bar{G})| > 3p-6$. Hence by Corollary 6.1.1, G is non-planar or $\bar{G}$ is non-planar.

6.15. Suppose G is a planar graph. Then

$$q \leq 3p-6.$$

Now, Average degree in G

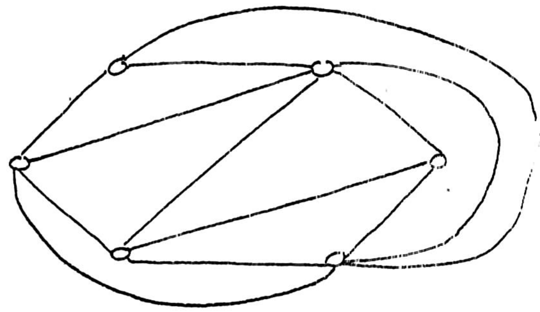
$$= \frac{\text{sum of all the degrees in } G}{\text{total no. of vertices in } G}$$

$$= \frac{2q}{p} \leq \frac{2.(3p-6)}{p}$$

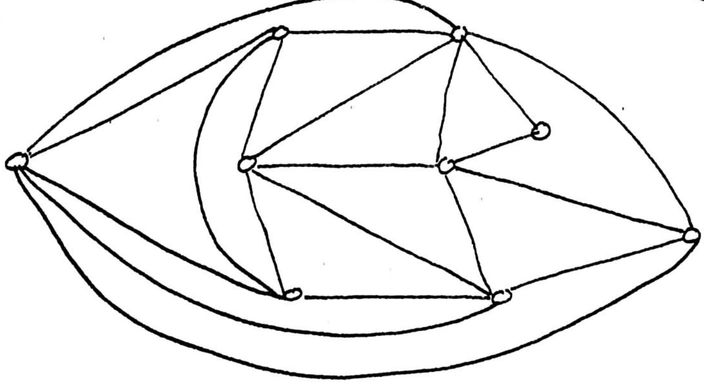
$$= \frac{6p-12}{p} = 6 - \frac{12}{p} < 6,$$ as required.

6.16 Both of the graphs are planar. Using the DMP algorithm we can get the two planar embeddings shown below.

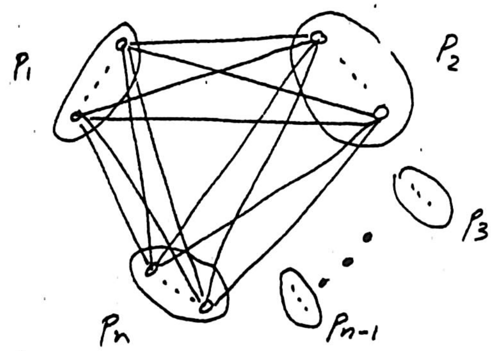
(a)



(b)



8.4 $K_{p_1,\dots,p_n}$ is the n-partite complete graph. It consists of n groups of vertices of sizes $p_1,\dots,p_n$.



No edges are allowed between vertices in the same group — but all other possible edges are allowed. If we color the p_1 vertices with color 1, the p_2 vertices with color 2, ..., and the p_n vertices with color n, we will get a legal coloring of $K_{p_1,\dots,p_n}$. So $\chi(K_{p_1,\dots,p_n}) \leq n$. But because we have all the other possible edges mention, $K_{p_1,\dots,p_n}$ cannot be legally colored with less than n colors. So

$$\chi(K_{p_1,\dots,p_n}) = n.$$

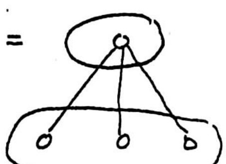
8.5 Hint: Just color each part with a different color. This will will gives us a legal coloring of G because G is k-partite. Hence

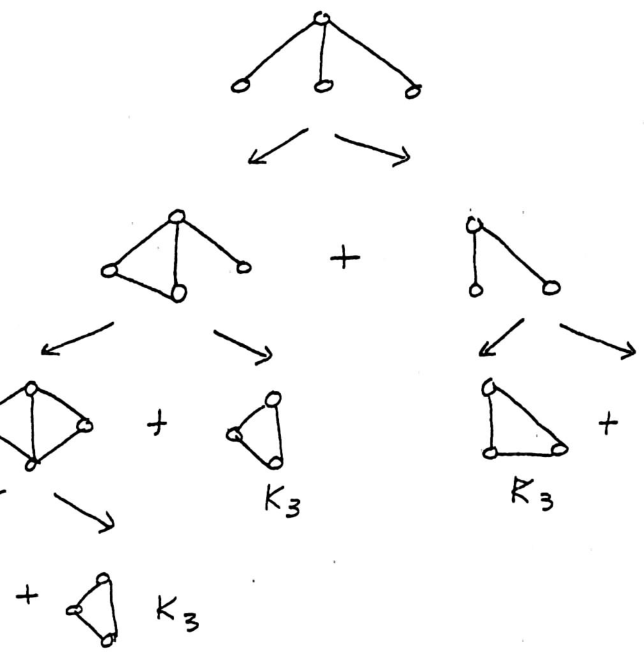
$$\chi(G) \leq k$$

8.10 Hint: Let V_i = set of vertices receiving color i $(i=1,\dots,k)$. Then $V_1,\dots,V_k$ will give us a k-partite graph because no edges can be between vertices

8.10 of the same group (because they received the same color). Now we can add all the other possible allowable edges to get a complete k-partite graph G'. Since G' was formed by adding edges to G, G is a subgraph of G' and so we get the required result.

8.27 (a)

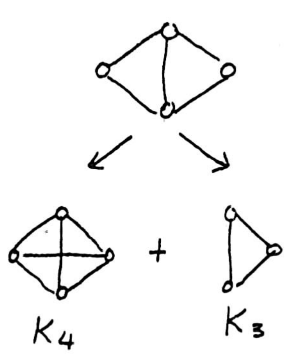
$K_{1,3}$ = 



$$\therefore\ P(K_{1,3},\lambda) = P(K_4,\lambda) + 3P(K_3,\lambda) + P(K_2,\lambda)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + 3\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)$$
$$= \lambda(\lambda-1)[\lambda^2-5\lambda+6+3\lambda-6+1] = \lambda(\lambda-1)^3$$

No. of 5-colorings $= P(K_{1,3},5) = 5(5-1)^3 = 320$

(b)



$$P(K_4-e,\lambda) = P(K_4,\lambda) + P(K_3,\lambda)$$
$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)(\lambda-2)^2$$

No. of 5-colorings
$= P(K_4-e,5) = 5(5-1)(5-2)^2 = 180.$