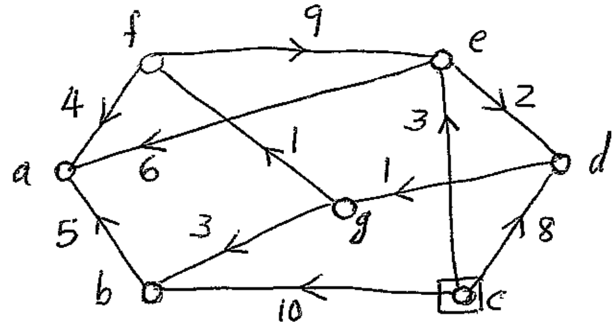
MAD 3301 - GRAPH THEORY　　　　　　　　　　　　　　FLORIDA INT'L UNIV.

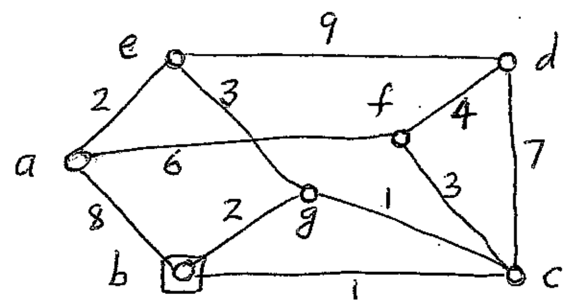
TEST #1 - FALL 2024　　　　　　　　　　　　　　　　　　TIME: 75 min.

Answer *all 6* questions. ***No calculators, cell-phones , or notes are allowed.*** *An unjustified answer will receive little or no credit.* BEGIN EACH OF THE 6 QUESTION ON 6 SEPARATE PAGES.

(15) 1. Find the *distances* from c to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm*.



(20) 2 (a) Find a *graph* with degree sequence $\langle 5,3,3,2,2,1 \rangle$ by using *Graphical Sequence Algorithm*.
(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* and starting at b.



(20) 3 (a) Find the *tree* that corresponds to the sequence $\langle 6, 4, 2, 6 \rangle$ via the *Prufer's Tree Decoding Algorithm*.
(b) The five characters *a, b, c, d, e* occur with frequencies *8, 12, 7, 20, 5;* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm*.

(15) 4 (a) Define what is a ***tree*** and what is a ***spanning-tree*** of a connected graph G.
(b) Prove that in any tree $T = \langle V(T), E(T) \rangle$ *we have* $|E(T)| = |V(T)| - 1$.

(15) 5 (a) Define what is the ***distance****, d(u,v),* from *u* to *v* in a *weighted digraph G.*
(b) If G is a ***disconnected graph****,* prove that $d(u,v) \le 2$ *for any* $u, v \in G^c$.

(15) 6 (a) Define what it means for the graph G to be ***isomorphic*** to the *graph H.*
(b) A ceratin tree T has 6 vertices of degree 5, 10 of degree 4, 9 of degree 3, and the rest of degree 1 or 2. What is the smallest possible value of $|V(T)|$?
[*You may use any theorems* ***proved in class*** *to answer question #6.*]

MAD 3301 - Graph Theory Florida Int'l Univ.

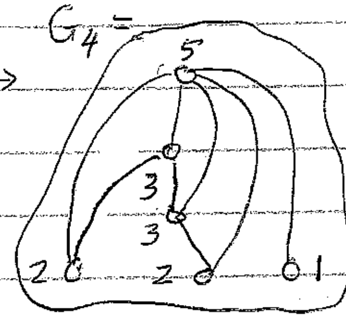
Solutions to Test #1 Fall 2024

#1.

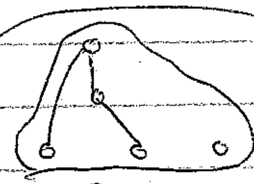
$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	$T(i)$	i	$V_0(i)$
∞	∞	$\underline{0}$	∞	∞	∞	∞	$\{a,b,c,d,e,f,g\}$	0	$c \to b,d,e$
∞	10	.	8	$\underline{3}$	∞	∞	$\{a,b,d,e,f,g\}$	1	$e \to a,d$
9	10	.	$\underline{5}$	.	∞	∞	$\{a,b,d,f,g\}$	2	$d \to g$
9	10	.	.	.	∞	$\underline{6}$	$\{a,b,f,g\}$	3	$g \to b,f$
9	9	.	.	.	$\underline{7}$	.	$\{a,b,f\}$	4	$f \to a,e$
$\underline{9}$	9	.	.	.	.	.	$\{a,b\}$	5	$a \to \emptyset$
.	$\underline{9}$	.	.	.	.	.	$\{b\}$	6	$b \to a$

$d(c,\cdot) =$ 9 9 0 5 3 7 6

#2 (a) $\langle 5, \underline{3,3,2,2,1} \rangle \longrightarrow G_4 =$

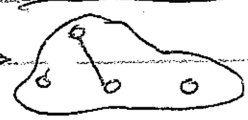


$\langle 2, \underline{2,1},1,0 \rangle \longrightarrow$



$= G_3$

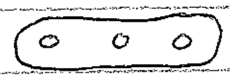
$\langle 1,0,1,0 \rangle \longrightarrow$



$= G_2$

$\langle 1, \underline{1}, 0\ 0 \rangle$

$\langle 0,0,0 \rangle \longrightarrow$

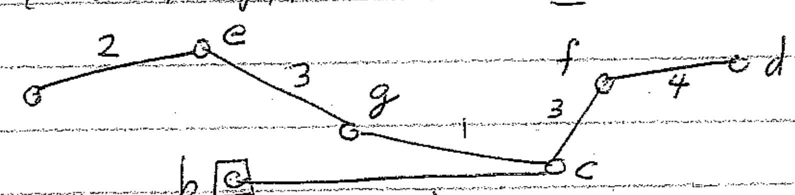


$= G_1$

(b)

$E(T)$	$V(T)$	a	b	c	d	e	f	g	i	$X_0(i)$
$\emptyset$	$\{b\}$	∞	$\underline{0}$	∞	∞	∞	∞	∞	0	$b \to a,c,g$
$\{\overline{bc}\}$	$\{b,c\}$	8	.	$\underline{1}$	∞	∞	3	2	1	$c \to d,f,g$
$\{\overline{bc},\overline{cg}\}$	$\{b,c,g\}$	8	.	.	7	3	3	$\underline{1}$	2	$g \to e$
$\{\overline{bc},\overline{cg},\overline{eg}\}$	$\{b,c,e,g\}$	8	.	.	7	$\underline{3}$	3	.	3	$e \to a$
$\{\overline{bc},\overline{cg},\overline{eg},\overline{ae}\}$	$\{a,b,c,e,g\}$	$\underline{2}$	.	.	7	.	3	.	4	$a \to f$
$\{\overline{bc},\overline{cg},\overline{eg},\overline{ae},\overline{cf}\}$	$\{a,b,c,e,f,g\}$	.	.	.	7	.	$\underline{3}$	.	5	$f \to d$
$\{\overline{bc},\overline{cg},\overline{eg},\overline{ae},\overline{cf},\overline{fd}\}$	$\{a,b,c,d,e,f,g\}$	.	.	.	$\underline{4}$	.	.	.	6	$d \to$ nothing new

$T_{min} =$



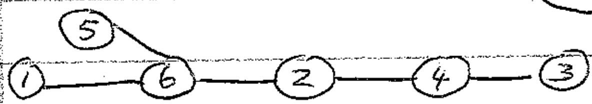
$\omega(T_{min}) = 1 + 1 + 3 + 2 + 3 + 4 = \boxed{14}$ (not requested this time)

$|\underline{s}| = p-2$, so $4 = p-2 \Rightarrow p = 6$.

#3 (a)

$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$\ell(i)$ — $s(i)$
1	2	1	2	1	3	1	1 — 6
0	2	1	2	1	2	2	3 — 4
0	2	0	1	1	2	3	4 — 2
0	1	0	0	1	2	4	2 — 6
0	0	0	0	1	1	5	5 — 6

T =



(b)

e, c, a, b, d

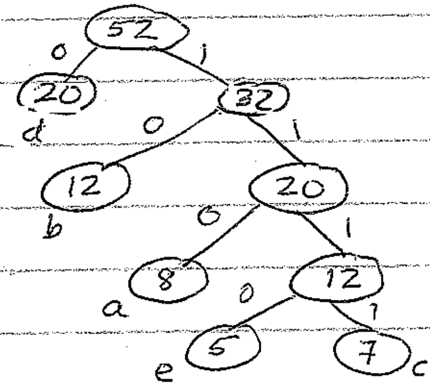
5, 7, 8, 12, 20

8, 12, 12, 20

12, 20, 20

20, 32

52



Char	a	b	c	d	e
Code	110	10	1111	0	1110
Code length	3	2	4	1	4
Freq.	8	12	7	20	5

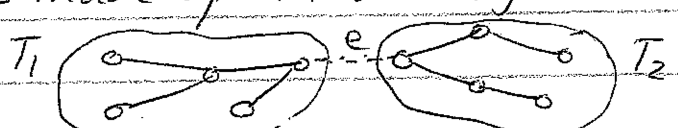
$WPL = 3(8) + 2(12) + 4(7) + 1(20) + 4(5) = \boxed{116}$

#4 (a) A tree is a non-empty connected graph which has no cycles. A spanning tree of G is any subgraph H of G which is a tree and has $V(H) = V(G)$.

(b) Let $p = |V(T)|$. We will prove the result by induction on p.

Basis: If $p = 1$, then $T \cong K_1$. So $E(T) = \emptyset$. Thus $|E(T)| = 0 = 1-1 = |V(T)|-1$. ~~Hence the result is true for $p=1$.~~

Ind. Step: Suppose the result is true for all trees with $\leq p$ vertices. Let T be any tree with $p+1$ vertices. Choose any edge e in T and consider the graph $T - \{e\}$. This will be a disconnected graph which is made up of two disjoint trees T_1 & T_2. So



$|E(T)| = |E(T_1)| + |E(T_2)| + |\{e\}|$

$= [|V(T_1)| - 1] + [|V(T_2)| - 1] + 1 = |V(T_1)| + |V(T_2)| - 1 = |V(T)| - 1.$

So if the result is true for all trees with $\leq p$ vertices, it will be true for all trees with $p+1$ vertices. The result now follows by Math. Induction.

#5 (a) $d(u,v) = \begin{cases} \text{length of the shortest directed path from } u \text{ to } v \text{ in } G, \\ +\infty \quad \text{if there is no directed path from } u \text{ to } v \text{ in } G. \end{cases}$

(b) Since G is disconnected, it has at least 2 connected components. There are 3 cases: Case(i): $u=v$. In this case $d_{G^c}(u,v) = 0 \le 2$.
Case(ii): u & v are from different components of G. In this case $\overline{uv}$ will be an edge in $E(G^c)$. So $d_{G^c}(u,v) = 1 \le 2$.
Case(iii) u & v are from the same component of G. In this case choose a vertex w in another component of G. Then $\overline{uw}$ & $\overline{wv}$ be edges in $E(G^c)$. So $u - w - v$ will be a path of length 2 from u to v in G^c. Thus $d_{G^c}(u,v) = 2 \le 2$ in this case.
So we will always have $d_{G^c}(u,v) \le 2$ for any $u, v \in V(G^c)$.

#6 (a) $G \cong H$ if there exists a bijection $\alpha : V(G) \to V(H)$ such that $\langle u,v\rangle \in E(G) \Leftrightarrow \langle \alpha(u), \alpha(v)\rangle \in E(G)$.

(b) Let $p = |V(T)|$, k = no. of vertices of degree 2 in T, and l = no. of vertices of degree 1 in T. Then

$$p = 6 + 10 + 9 + k + l = 25 + k + l$$

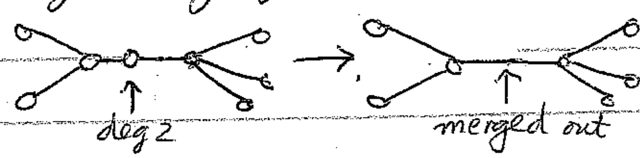
Also $2 \cdot |E(T)| = 2(p-1)$ = sum of the degrees in T.

So $2(25 + k + l - 1) = 6(5) + 10(4) + 9(3) + k(2) + l(1)$

$\therefore \quad 50 + \cancel{2k} + 2l - 2 = 30 + 40 + 27 + \cancel{2k} + l$

$\therefore \quad l = 70 + 27 + 2 - 50 = 99 - 50 = 49.$

Now k can be anything but by merging-out all the vertices of degree 2, we can get a tree T with the least no. of vertices,



namely with $\underset{\text{deg 5}}{6} + \underset{\text{deg 4}}{10} + \underset{\text{deg 3}}{9} + 0 + \underset{\text{deg 1}}{49} = 25 + 49 = \boxed{74}$

So the smallest number of vertices T can have is 74.

END