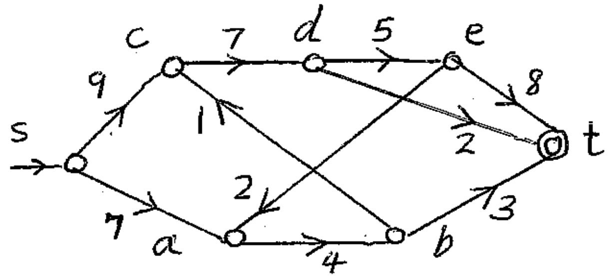
MAD 3301 - GRAPH THEORY FLORIDA INT'L UNIV.

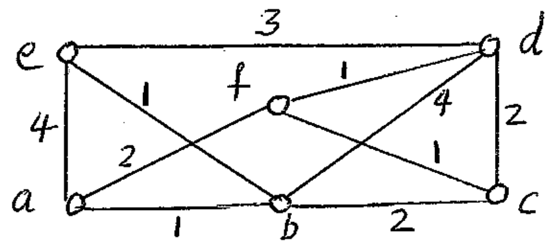
TEST #2 – FALL 2024 TIME: 75 min.

Answer *all 6* questions. ***No calculators, notes, or on-line data are allowed.*** *An unjustified answer will receive little or no credit.* ***Draw a line to separate each of the 6 solutions to the 6 questions.***

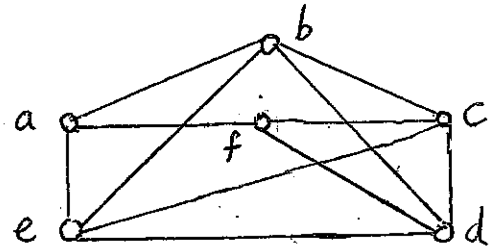
(15) 1. Find a *maximal flow* f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the *source-separating set of vertices* S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

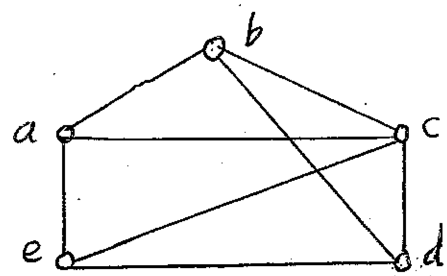


(16) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(24) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm.*

(b) Define what is a *legal-coloring* of a graph G; & prove that if G has *no odd cycles* then $\chi(G) \leq 2$.



(15) 5(a) Define what is a *minimum postman walk* and define what is a *minimum salesman walk* in a weighted multi-graph G.

(b) Write down *Ore's theorem* & use it to prove that any graph G with deg(x)+deg(y) $\geq$ |V(G)| - 1 for all pairs of *non-adjacent* vertices x & y, has a *Hamilton path*.

(15) 6(a) Define what is the *dual* of *G* with respect to *a planar embedding* $\mathcal{E}$ of a planar graph G and define what is a *self-dual* graph.

(b) Let $\mathcal{E}$ be a planar embedding of a *connected planar-graph* G in which each region is bounded by at least 8 edges. Prove that $3q \leq 4(p-2)$.

[*You may use any theorem that was proved in class for Qu.#6, if needed.*] END

MAD 3301 – Graph Theory — Florida Int'l Univ.
Solutions to Test #2 — Fall 2024

#1 1st aug. semi-path:

$s \xrightarrow{(0,7)} a \xrightarrow{(0,4)} b \xrightarrow{(0,3)} t$

slacks 7 4 3 ($\mu_1 = 3$)

2nd aug. semi-path:

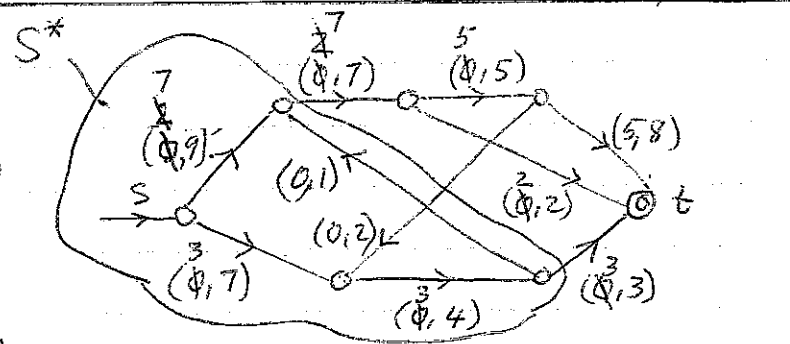
$s \xrightarrow{(0,9)} c \xrightarrow{(0,7)} d \xrightarrow{(0,2)} t$

slacks 9 7 2 ($\mu_2 = 2$)

3rd aug. semi-path

$s \xrightarrow{(2,9)} c \xrightarrow{(2,7)} d \xrightarrow{(0,5)} e \xrightarrow{(0,8)} t$

slacks 7 5 5 8 ($\mu_3 = 5$)



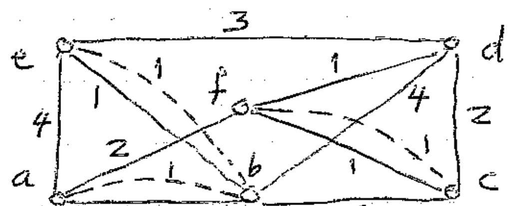
$S^* = \{u \in V(G) : \text{flow can be sent from } s \text{ to } u\}$
$= \{s, a, b, c\}$ $\quad c(S^*) = 7 + 3 = \boxed{10}$

$Val(f^*) = $ net flow into t
$= 5 + 2 + 3 = \boxed{10}$ ✓

#2

	a	c	e	f
a	·	3	2	2
c	·	·	3	1
e	·	·	·	4
f	·	·	·	·

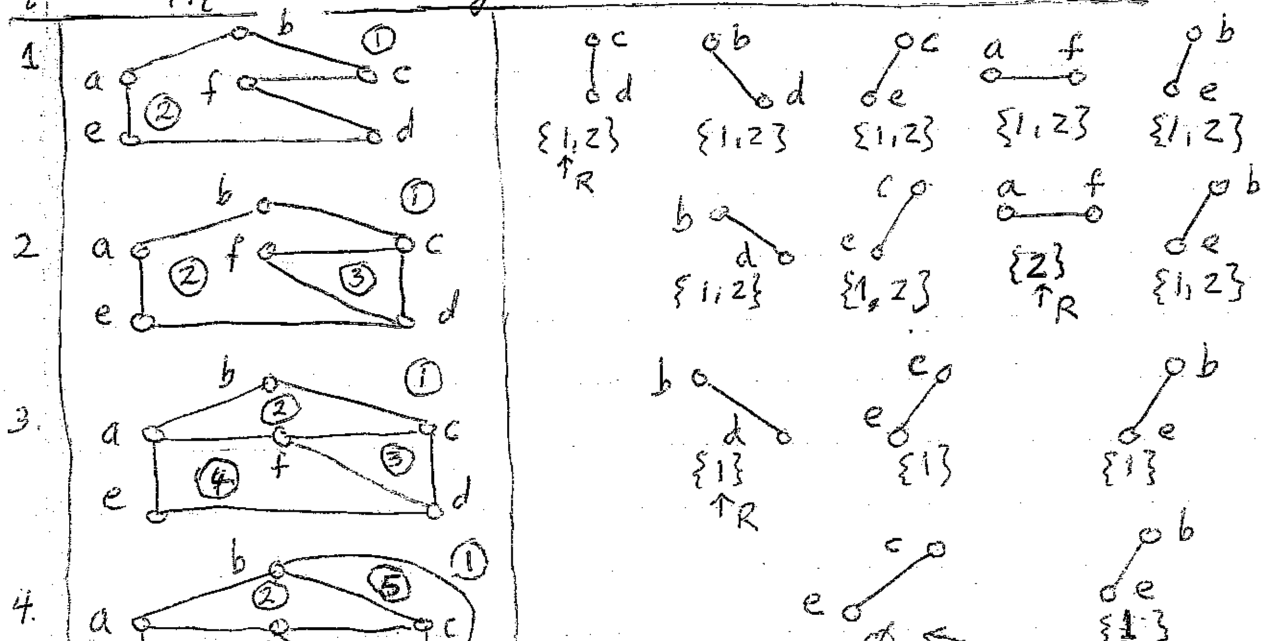
$\{a,c\} + \{e,f\}$: $3 + 4 = 7$ $\quad$ $\{a,e\} + \{c,f\}$: $2 + 1 = 3$ ✓ $\quad$ $\{a,f\} + \{c,e\}$: $2 + 3 = 5$



Mininimum postman walk W is:

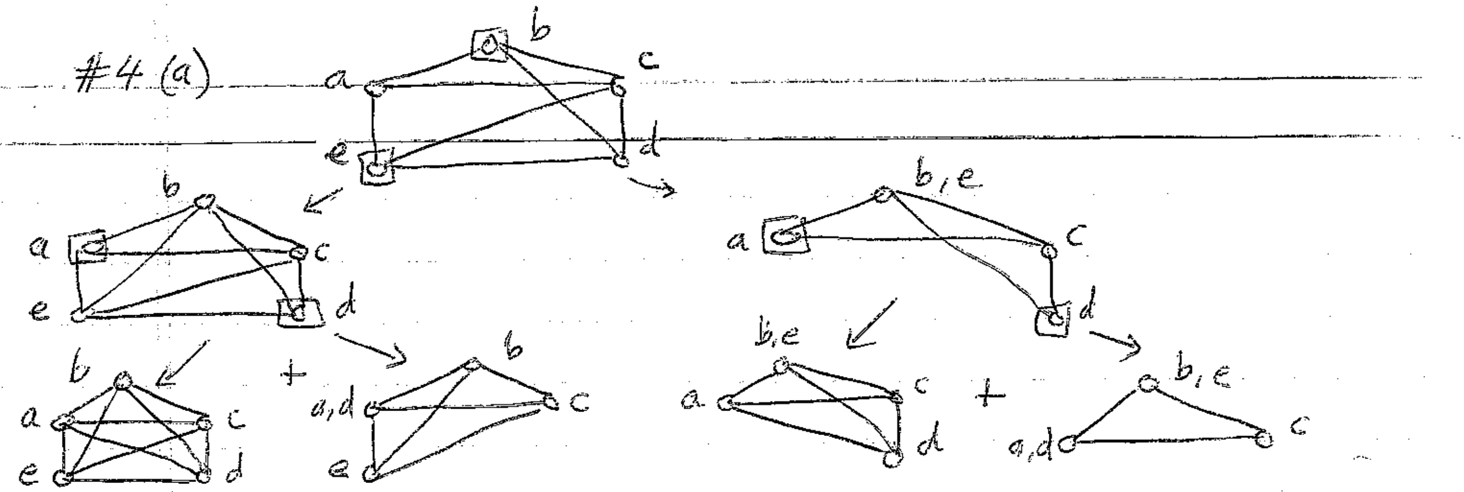
$a \overset{1}{—} b \overset{1}{—} a \overset{4}{—} e \overset{1}{—} b \overset{1}{—} e \overset{3}{—} d \overset{2}{—} c \overset{1}{—} f \overset{1}{—} c \overset{2}{—} b \overset{4}{—} d \overset{1}{—} f \overset{2}{—} a$ $\quad$ length(W) = 24.

#3 i $\quad H_i$ $\quad$ Segments of G relative to H_i



G is NON-PLANAR

#4 (a)



$$P_G(\lambda) = P_{K_5}(\lambda) + 2P_{K_4}(\lambda) + P_{K_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3)$$
$$+ \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda-2)\left[(\lambda-3)(\lambda-4) + 2(\lambda-3) + 1\right] = \lambda(\lambda-1)(\lambda-2)\cdot[\lambda^2-5\lambda+7]$$

4(b) By deleting the edges from the cycles in G (one at time), we will end up with a spanning forest $F = T_1 \cup T_2 \cup \cdots \cup T_k$ of G. Now choose a vertex v_i in T_i & let it be the root of T_i (for $i=1$ to k). Then color the even levels of each T_i with color #1 & the odd levels of T_i with color #2. If we add back the edges we took out, we will see that each edge joins a color #1 to a color #2 vertex (otherwise, we would get an odd cycle in G). Hence this is a legal color of G. So $\chi(G) \le 2$.

Def A legal coloring of G is a function $f: V(G) \to \mathbb{Z}^+$ such that $f(u) \neq f(v)$ whenever u & v are adjacent in G.

#5 (a) A minimum postman walk of G is any closed walk of G which includes each edge of G (at least once) & is of the smallest possible total length. A minimum salesman walk of G is any closed walk of G which includes each vertex of G (at least once) & is of the smallest possible total length.

5(b) Ore's Theorem: In any graph G with $|V(G)| = p \geq 3$, if we have $\deg(x) + \deg(y) \geq p$ for all pairs of non-adj vertices, then G has a Hamilton cycle.

First of all, if $p = |V(G)| = 1$, then the empty path $\langle v_1 \rangle$ is a Hamilton cycle of G. So suppose $p = |V(G)| \geq 2$. Let H be the graph obtained from G by adding a new vertex v_{p+1} & edges from v_{p+1} to $v_1, \dots, v_p$.

#5(b) Since $\deg_G(x) + \deg_G(y) \geq p-1$ for any pair of non-adjacent vertices x & y in G, it follows that $\deg_H(x) + \deg_H(y) \geq (p-1)+1+1$ $= p+1$, for all pairs of non-adj. vertices in H. Since $p+1 \geq 3$, it follows from Ore's Theorem that H has a Hamilton cycle, C. Now if we remove the vertex v_{p+1} from C, we will get a Hamilton path in G.

#6(a) The <u>dual</u> of G with respect to $\mathcal{E}$ is the multi-graph $G^*_{\mathcal{E}} = \langle V(G^*_{\mathcal{E}}), E(G^*_{\mathcal{E}})\rangle$ where $V(G^*)$ = the set of regions into which $\mathcal{E}$ divides the plane & for each edge that two regions R_1 & R_2 share we get an edge joining R_1 & R_2 in $E(G^*_{\mathcal{E}})$. A planar graph G is <u>self-dual</u> if $G^*_{\mathcal{E}} \cong G$ for every planar embedding $\mathcal{E}$ of G.

6(b) Let $A_1, A_2, \ldots, A_r$ be the regions into which $\mathcal{E}$ divides the plane. Then $e(A_i) \geq 8$ for each $i = 1, \ldots, r$. [Here $e(A_i)$ is the number of edges of the region A_i.]. So

$$8r \leq e(A_1) + e(A_2) + \cdots + e(A_r) \doteq 2 \cdot q$$

because each edge is counted twice. Thus $4r \leq q$.
Since G is a connected planar graph, it follows from Euler's Planarity Theorem that $r = q + 2 - p$. So

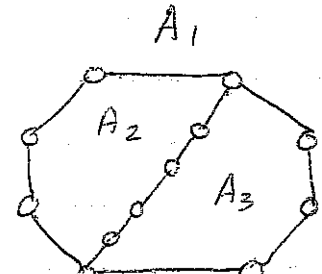
$$4(q+2-p) \leq q.$$

$\therefore \quad 4q + 8 - 4p \leq q \qquad \therefore \quad 3q \leq 4p - 8 = 4(p-2).$

Thus $3q \leq 4(p-2)$.

<u>END</u>

This is not a part of the solution.
It is just an example to show what can happen. Here we have a connected planar graph with <u>$p = 12$</u> and <u>$q = 13$</u>. Let us check that

$G =$ 

$e(A_1) = 8, \; e(A_2) = 9 = e(A_3)$

$$3q = \underbrace{3(13)}_{39} \leq 4(p-2) = \underbrace{4(10)}_{40}.$$

Yes! <u>$39 \leq 40$</u>.

Is $4r \leq q$? Yes!
$12 \leq 13$.