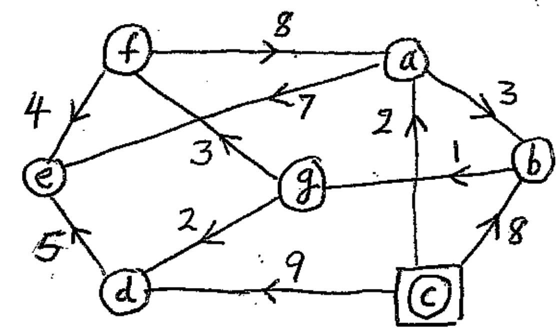
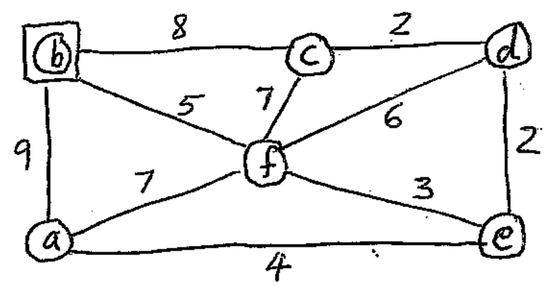
MAD 3301 - GRAPH THEORY | FLORIDA INT'L UNIV.
TEST #1 - SPRING 2025 | TIME: 75 min.

Answer *all 6* questions. ***No calculators, cell-phones, or notes are allowed.*** *An unjustified answer will receive little or no credit.* BEGIN EACH OF THE 6 QUESTION ON 6 SEPARATE PAGES.

(15) 1. Find the *distances* from *c* to each of the other vertices of the graph on the right by using *Dijkstra's Algorithm.*



(20) 2 (a) Find a *graph* with degree sequence ⟨5,3,3,3,2,2⟩ by using *Graphical Sequence Algorithm.*
(b) For the graph on the right, find a *minimal spanning tree* by using *Prim's Algorithm* and starting at *b.*



(20) 3 (a) Find the *tree* that corresponds to the sequence ⟨5, 3, 2, 5⟩ via the *Prufer's Tree Decoding Algorithm.*
(b) The five characters *a, b, c, d, e* occur with frequencies *10, 12, 6, 20, 4;* respectively. Find an *optimal binary coding* for these five characters and the *weighted-path length* of your coding by using *Huffman's algorithm.*

(15) 4 (a) Define what is the ***adjacency matrix*** *A* of a **graph** with V(G) = {1,2,3,…, p}.
(b) Prove that $A^n[i, j]$ = the number of *walks of length n* from *i* to *j* in G.

(15) 5 (a) Define what's a ***tree,*** & define what's a ***spanning-tree*** of a *connected* graph G.
(b) If G is a *disconnected graph* with p vertices, prove that E(G) ≤ (p-1).(p-2) / 2.

(15) 6 (a) Define what's a ***legal flow*** & the ***value*** *of a legal flow* in a network $N = \langle G,s,t,c \rangle$.
(b) A *tree T* has 4 vertices of deg. 5, 5 of deg. 4, 6 of deg. 3, & the rest of degree 1 or 2. How many *leaves* does T have? What is the *smallest possible* value of |*E(T)*|?
[*You may use any theorems* ***proved in class*** *to answer question #6.*]

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Solutions to Test #1 — Spring 2025

#1.

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$	$T(i)$	i	$V_0(i) \rightarrow$
∞	∞	$\underline{0}$	∞	∞	∞	∞	$\{a,b,\underline{c},d,e,f,g\}$	0	$c \rightarrow a,b,d$
$\underline{2}$	8	.	9	∞	∞	∞	$\{\underline{a},b,d,e,f,g\}$	1	$a \rightarrow b,e$
.	$\underline{5}$	.	9	9	∞	∞	$\{b,d,e,f,g\}$	2	$b \rightarrow g$
.	.	.	9	9	∞	$\underline{6}$	$\{b,d,e,\underline{f}\}$	3	$g \rightarrow d,f$
.	.	.	$\underline{8}$	9	9	.	$\{\underline{d},e,f\}$	4	$d \rightarrow e$
.	.	.	.	$\underline{9}$	9	.	$\{\underline{e},f\}$	5	$e \rightarrow \emptyset$
.	.	.	.	.	$\underline{9}$	.	$\{\underline{f}\}$	6	$f \rightarrow e$
$d(c,\cdot)=$ 2	5	0	8	9	9	6	$\emptyset$		

#2 (a) $\langle 5, \underline{3,3,3,2,2} \rangle$

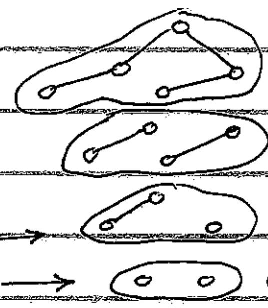
$\langle \underline{2,2},2,1,1 \rangle$

$\langle \overline{1,1},1,1 \rangle$

$\langle \overline{0},1,1 \rangle$

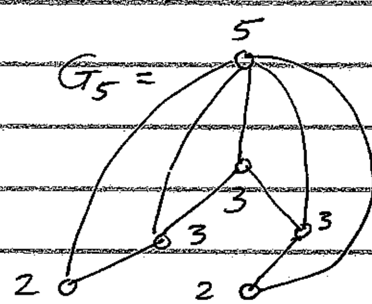
reorder $\langle 1,\underline{1},0 \rangle \longrightarrow$

$\langle \overline{0,0} \rangle \longrightarrow$



G_4, G_3, G_2, G_1

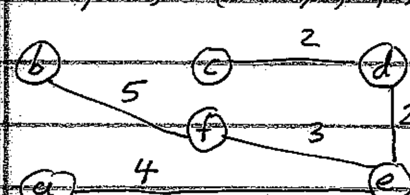
$G_5 =$



2(b)

$E(T)$	$V(T)$	a	b	c	d	e	f	i	$x_0(i)$
$\emptyset$	$\{b\}$	∞	$\underline{0}$	∞	∞	∞	∞	0	$b \rightarrow a,c,f$
$\{\overline{bf}\}$	$\{b,f\}$	9	.	8	∞	∞	$\underline{5}$	1	$f \rightarrow a,c,d,e$
$\{\overline{bf},\overline{fe}\}$	$\{b,f,e\}$	7	.	7	6	$\underline{3}$	.	2	$e \rightarrow a,d$
$\{\overline{bf},\overline{fe},\overline{ed}\}$	$\{b,f,e,d\}$	4	.	7	$\underline{2}$	.	.	3	$d \rightarrow c$
$\{\overline{bf},\overline{fe},\overline{ed},\overline{dc}\}$	$\{b,f,e,d,c\}$	4	.	$\underline{2}$	.	.	.	4	$c \rightarrow \emptyset$
$\{\overline{bf},\overline{fe},\overline{ed},\overline{dc},\overline{ea}\}$	$\{b,f,e,d,c,a\}$	$\underline{4}$	.	.	.	.	.	5	$a \rightarrow \emptyset$

$T_{min} =$



$\omega(T_{min}) = 5+3+2+2+4 = \boxed{16}$

[These are the underlined nos. above]

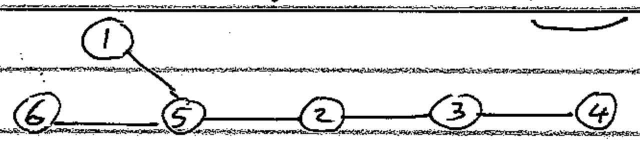
#3(a) $\underset{\sim}{s} = \langle 5,3,2,5 \rangle$. So $p-2 = |s| = 4$. $\therefore\ p = \boxed{6}$.

#3(a)

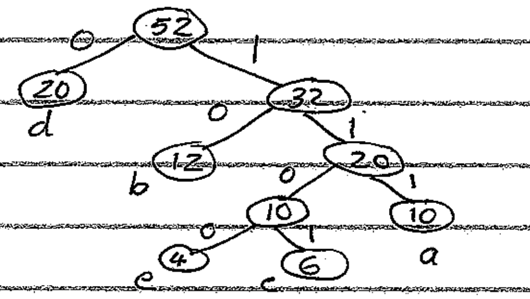
$d_i(1)$	$d_i(2)$	$d_i(3)$	$d_i(4)$	$d_i(5)$	$d_i(6)$	i	$l(i)$ — $s(i)$
1	2	2	1	3	1	1	1 — 5
0	2	2	1	2	1	2	4 — 3
0	2	1	0	2	1	3	3 — 2
0	1	0	0	2	1	4	2 — 5
0	0	0	0	1	1	5	5 — 6

$\underset{\sim}{S}$

join last 2 vertices of deg. 1

$T =$ 

3(b)
e c a b d
4, 6, 10, 12, 20
10, 10, 12, 20
12, 20, 20
20, 32
52



Char	a	b	c	d	e
Code	111	10	1101	0	1100
length	3	2	4	1	4
Freq.	10	12	6	20	4

W.P.L (coding)
$= 3(10) + 2(12) + 4(6) + 1(20) + 4(4)$
$= 30 + 24 + 24 + 20 + 16 = \boxed{114}$

#4(a) The adjacency matrix of G is the $p \times p$ matrix A_G defined by $A_G[i,j]$ = number of edges from vertex i to vertex j.

(b) We shall prove the result by Mathematical induction on n.

Basis: If $n=1$, then by definition we have $A^1[i,j] = A[i,j]$ = no. of edges from i to j = no. of walks of length 1 from i to j. So the result is true for $n=1$, and for all i & all j.

Ind. step: Suppose the result is true for walks of length n. Then $(A^n)[i,j]$ = no. of walks of length n from i to j for all i & all j. Now

$$\begin{pmatrix}\text{No. of walks of length}\\ n+1 \text{ from } i \text{ to } j \text{ in } G\end{pmatrix} = \sum_{k=1}^{p} \begin{pmatrix}\text{no. of walks of length}\\ n \text{ from } i \text{ to } k \text{ in } G\end{pmatrix} \cdot \begin{pmatrix}\text{no. of walks of length}\\ 1 \text{ from } k \text{ to } j \text{ in } G\end{pmatrix}$$

$$= \sum_{k=1}^{p} (A^n)[i,k] \cdot A^1[k,j] = (A^{n+1})[i,j]$$ by the definition of matrix multiplication. So if the result is true for walks of length n from i to j, it will be true for walks of length $n+1$ from i to j.

Conclusion: By the Principle of Mathematical Induction, it follows that the result is true for walks of any length n from any i to any j.

#5(a) A tree is any non-empty, connected graph which contains no non-trivial cycles. A spanning tree of a connected graph G is any subgraph T of G such that T is a tree and $V(T) = V(G)$.

(b) Suppose G is a disconnected graph with p vertices. Then we can split G into two parts G_1 & G_2 such that $G_1 \cup G_2 = G$ and there are no shared edges or vertices between G_1 & G_2. Let $k = |V(G_1)|$. Then $|V(G_2)| = p-k$ and $|E(G)| \leq k(k-1)/2 + (p-k)(p-k-1)/2$.

$$\therefore \frac{(p-1)(p-2)}{2} - |E(G)| \geq \left[(p-1)(p-2) - k(k-1) - (p-k)(p-k-1)\right]/2$$
$$= \left[p^2 - 3p + 2 - k^2 + k - p^2 - k^2 + 2pk + p - k\right]/2$$
$$= \left[2pk - 2k^2 - 2p + 2\right]/2$$
$$= pk - k^2 - p + 1 = \underbrace{(k-1)}_{\geq 0}\underbrace{(p-k-1)}_{\geq 0} \geq 0.$$

because $k \geq 1$ & $(p-k) \geq 1$. Hence

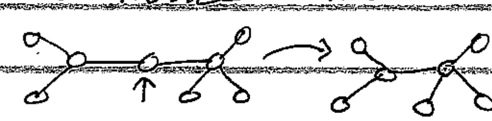
$(p-1)(p-2)/2 - |E(G)| \geq 0$ and so $|E(G)| \leq (p-1)(p-2)/2$.

#6 (a) A legal flow in a network $N = \langle G, s, t, c \rangle$ is any function $f: E(G) \to [0, \infty)$ such that $f(e) \leq c(e)$ for each $e \in E(G)$ and such that $\sum_{e \in In(v)} f(e) = \sum_{e \in Out(v)} f(e)$ for each $v \in V(G) - \{s, t\}$. The value of a legal flow f is defined by $Val(f) = \sum_{e \in In(t)} f(e) - \sum_{e \in Out(t)} f(e)$.

6(b). Let $p = |V(T)|$, ℓ = no. of leaves in T and k = no. of vertices of degree 2 in T. Then $p = 4 + 5 + 6 + k + \ell = 15 + k + \ell$ and $|E(T)| = p - 1 = 14 + k + \ell$. Now by a Theorem proved in class we know sum of degrees in $T = 2 \cdot |E(T)|$. So

$$4(5) + 5(4) + 6(3) + k(2) + \ell(1) = 2 \cdot |E(T)|$$
$$\therefore \; 20 + 20 + 18 + 2k + \ell = 2(14 + k + \ell) = 28 + 2k + 2\ell$$
$$\therefore \; 58 = 28 + \ell$$

So $\ell = 58 - 28 = 30$. Hence T has 30 leaves. By merging-out all the vertices of degree 2, we can make $k = 0$.



So min. value of $|E(T)| = 14 + 0 + \ell = 14 + 30 = \boxed{44}$