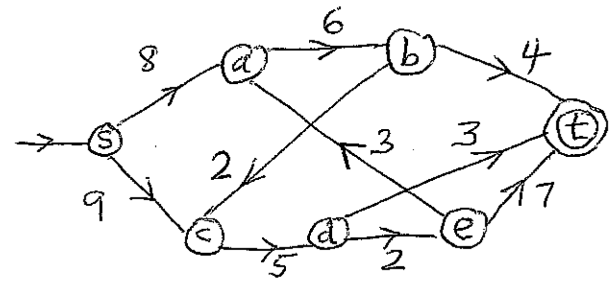
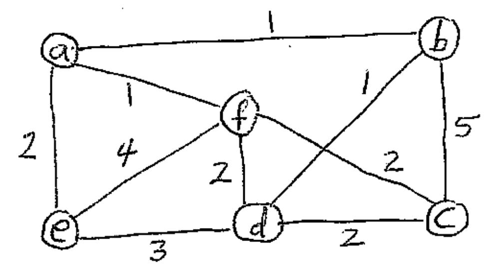
MAD 3301 - GRAPH THEORY FLORIDA INT'L UNIV.
TEST #2 - SPRING 2025 TIME: 75 min.

Answer *all 6* questions. ***No calculators, notes, or on-line data are allowed.*** *An unjustified answer will receive little or no credit.* ***Draw a line to separate each of the 6 solutions to the 6 questions.***

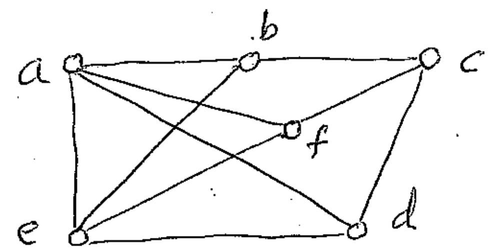
(15) 1. Find a *maximal flow* f^* in the network on the right by using the *Ford-Fulkerson Algorithm*. Also find the *source-separating set of vertices* S^* corresponding to f^*.



(15) 2. Find a *minimum postman walk* of the graph on the right by using the *Postman Algorithm*; and find *the total length* of your minimum postman walk?

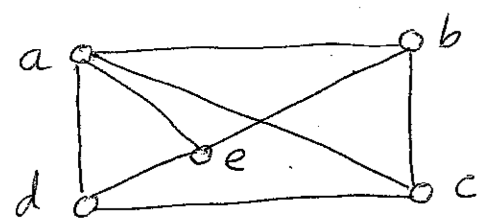


(16) 3. Determine whether or not the graph on the right is planar by using the *DMP Planarity Algorithm*. [*Show the embeddings for each step of the algorithm.*]



(24) 4(a) Find $P_G(\lambda)$ for the graph G on the right by using the *Chromatic Polynomial Algorithm.*

(b) Prove that $P_T(\lambda) = \lambda.(\lambda-1)^{n-1}$ for any tree T with n vertices.



(15) 5(a) Define what is a ***minimum postman walk*** and define what is a ***minimum salesman walk*** of a *weighted multi-graph* G.

(b) Suppose G is a graph with p vertices, $p \geq 3$, and for any pair of non-adjacent vertices, x & y, deg(x) + deg(y) $\geq p$. Let $\langle v_1, v_2, \ldots, v_n \rangle$ be any ***maximal path*** in G. Prove that the vertices $\langle v_1, v_2, \ldots, v_n, v_1 \rangle$ can be *rearranged* to form a *cycle*.

(15) 6(a) Define what is a ***planar graph G*** and define what is the ***dual of G*** *with respect to a planar embedding* $\mathcal{E}$ of G.

(b) Let $\mathcal{E}$ be a planar embedding of a *connected planar-graph* G in which each region is bounded by ***at least*** *12 edges*. Prove that $5.q \leq 6.(p-2)$.

[*You may use any theorem that was proved in class for Qu.#6, if needed.*] END

MAD 3301 - Graph Theory   Florida Int'l Univ.

Solutions to Test #2   Spring 2025

#1 1st Aug. semi-path

$s \xrightarrow{(0,8)} a \xrightarrow{(0,6)} b \xrightarrow{(0,4)} t$

Slacks: 8 6 4 $\mu_1 = 4$

2nd Aug. semi-path

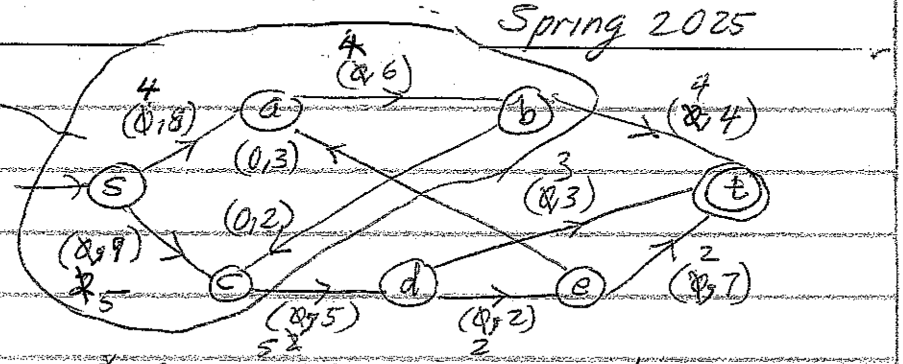
$s \xrightarrow{(0,9)} c \xrightarrow{(0,5)} d \xrightarrow{(0,2)} e \xrightarrow{(0,7)} t$

Slacks 9 5 2 7 $\mu_2 = 2$

3rd Aug. semi-path

$s \xrightarrow{(2,9)} c \xrightarrow{(2,5)} d \xrightarrow{(0,3)} t$

slacks 7 3 3 $\mu_3 = 3$

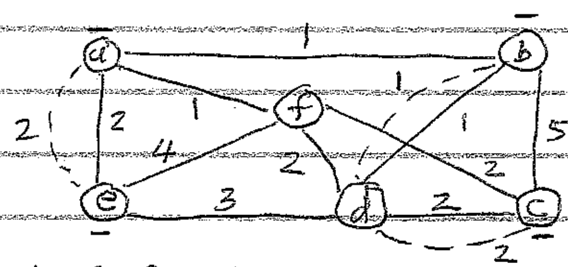


$S^* = \{u \in V(G) : \text{flow can be sent from } s \text{ to } u\}$
$= \{s, a, b, c\}$ $c(S^*) = 5 + 4 = \boxed{9}$

$Val(f^*) = \text{net flow into } t = 4+3+2 = \boxed{9}$

#2

$d(\cdot,\cdot)$	a	b	c	e
a	·	1	3	$\underline{2}$
b	·	·	$\underline{3}$	3
c	·	·	·	5
e	·	·	·	·



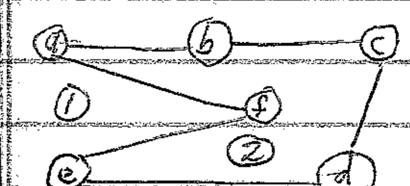
$\{a,b\}+\{c,e\}$ $\{a,c\}+\{b,e\}$ $\{a,e\}+\{b,c\}$ ✓

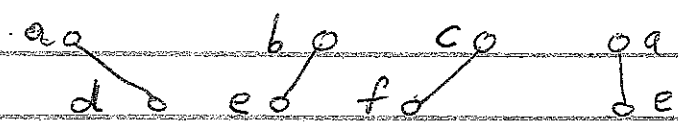
The minimum postman walk is: $1+5=6$ $3+3=6$ $2+3=\boxed{5}$

$a \overset{2}{—} e \overset{2}{—} a \overset{1}{—} b \overset{1}{—} d \overset{1}{—} b \overset{5}{—} c \overset{2}{—} d \overset{2}{—} c \overset{2}{—} f \overset{2}{—} d \overset{3}{—} e \overset{4}{—} f \overset{1}{—} a.$

#3   i H_i Segments of G relative to H_i

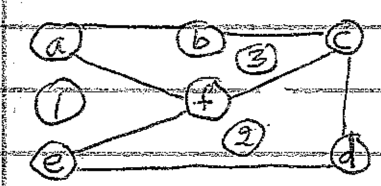
1

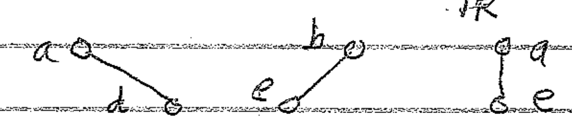




R_1: {1,2}   {1,2}   {1,2}   {1,2}
↑R

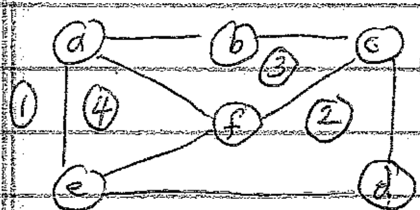
2.

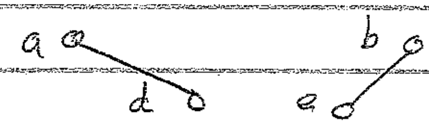




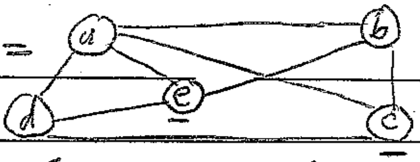
R_2: {1}   {1}   {1}
↑R

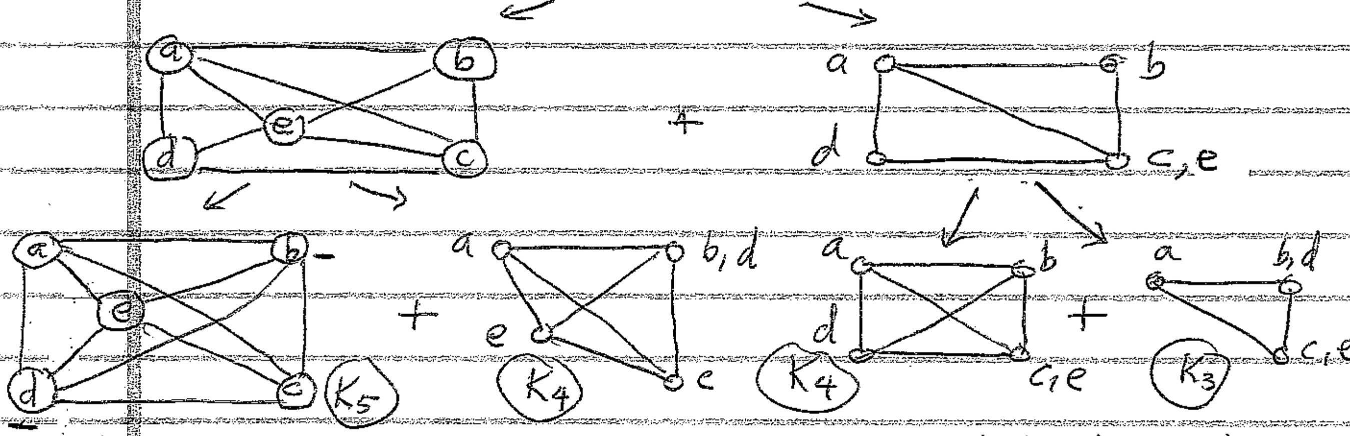
3.





R_3: {1} $\emptyset$ STOP. NON-PLANAR

4(a) $G=$ 



$\therefore P_G(\lambda) = P_{K_5}(\lambda) + 2P_{K_4}(\lambda) + P_{K_3}(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$
$+2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-3)$
$= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)(\lambda-4)+2(\lambda-3)+1] = \lambda(\lambda-1)(\lambda-2)[\lambda^2-5\lambda+7]$

(b) We will prove the result by induction on n. If $n=1$, then $T=K_1$ and $P_T(\lambda)=\lambda(\lambda-1)^{1-1}$. So the result is true for $n=1$. Suppose the result is true for all trees with n vertices. Let T be a tree with $(n+1)$ vertices. Then T has a leaf, v_0 say. (Just look at any maximal path in T, the endpoints will always be leaves.) Put $T' = T-\{v_0\}$. Then T' is a tree with with n vertices, so $P_{T'}(\lambda)=\lambda(\lambda-1)^{n-1}$. Hence $P_T(\lambda) = P_{T'}(\lambda)\cdot(\lambda-1)$ $=\lambda(\lambda-1)^{n+1-1}$ because there are $\lambda-1$ ways of coloring v_0. So if the result is true for n, it will true for $n+1$. By the Principle of Math Induction, it follows that the result is true for all trees.

#5 (a) A minimum postman walk of G is any closed walk of G which includes each edge of G (at least once) & is of the smallest possible total length. A minimum salesman walk of G is any closed walk of G which includes each vertex of G & is of the smallest possible total length

(b) First observe that if v_1 is adjacent to v_n, then $\langle v_1, v_2, \ldots, v_n, v_1\rangle$ will instantly form a cycle in G. So suppose v_1 is not adj. to v_n. Since the path was maximal, v_1 & v_k cannot be adjacent to any vertices outside of $\{v_1, v_2, \ldots, v_n\}$. We claim that there must be an $i \in \{2,3,\ldots,n-1\}$ such that $v_1v_i \in E(G)$ & $v_{i-1}v_n \in E(G)$. From this we will see that $\langle v_1, v_2, v_3, \ldots, v_{i-1}, v_n, v_{n-1}, v_{n-2}, \ldots, v_{i+1}, v_i, v_1\rangle$ will be a cycle in G. Now suppose there is no i such that $v_1v_i \in E(G)$ & $v_iv_n \in E(G)$. Then every time v_1 is adj. to vertex v_k, v_n cannot be adj. to the previous vertex v_{k-1}. So

$$\deg(v_n) \leq (n-1) - \deg(v_1) \leq (p-1) - \deg(v_1) \quad (\text{bec. } n \leq p).$$

Hence $\deg(v_1) + \deg(v_n) \leq p-1$ contrading $\deg(v_1) + \deg(v_n) \geq p$. So there is an $i \in \{2,3,\ldots,n-1\}$ with $v_1v_i \in E(G)$ & $v_{i-1}v_n \in E(G)$

#6.(a) A graph G is planar if we can draw it in the plane so that no two edges intersect, except possible at their endpoints. The dual of G (w.r.t. the embedding $\mathcal{E}$) is the multi-graph $G^* = \langle V(G^*), E(G^*)\rangle$ where $V(G^*)$ = set of regions into which $\mathcal{E}$ partitions $\mathbb{R}^2$ — and for each edge that two regions R_1 & R_2 share in $\mathcal{E}$, we get an edge between R_1 & R_2 in $E(G^*)$

(b) Let $A_1, A_2, \ldots, A_r$ be the regions into which the plane $\mathbb{R}^2$, is partitioned by $\mathcal{E}$. Let $e(A_i)$ = no. of edges of A_i. Then $12 \cdot r \leq e(A_1) + e(A_2) + \cdots + e(A_r) \leq 2q$ because $e(A_i) \geq 12$ for each i and each edge is counted as a boundary at most twice. So $6r \leq q$. But G is a connected planar graph, so $r = q + 2 - p$, by Euler's Planarity Theorem. Hence

$$6(q+2-p) \leq q. \quad \text{So } 6q + 12 - 6p \leq q$$

Thus $5q \leq 6p - 12 = 6(p-2)$ $\therefore$ $\boxed{5q \leq 6 \cdot (p-2)}$

END-