Chapter 10

Extremal Theory

Section 10.0 Introduction

We now begin a study of one of the most elegant and deeply developed areas in all of graph theory, extremal graph theory. We have often dealt with extremal questions. For example, earlier we tried to determine the minimum number of edges e so that every graph of order n with at least e edges contained a cycle. This was one of the first problems we attempted when studying trees. Other extremal problems we have investigated include ramsey theory, finding bounds on the cardinality of neighborhood unions that ensured hamiltonian cycles and degree requirements that ensured hamiltonian properties.

Extremal graph theory, in its most general form, concerns any problem which attempts to determine the relation between graph invariants (such as order, size or minimum degree) and a graph property (like being hamiltonian, containing a perfect matching or containing a particular subgraph G_1). Typically, given a graph property P, an invariant i and a class of graphs $\hat{H}$, one tries to determine the least value m such that every graph G in $\hat{H}$ with $i(G) > m$ has property P.

We shall limit our investigation to that question generally credited with starting extremal theory and to the beginnings of the research that sprang from this question. This study is rich in counting techniques and estimations. We shall use elementary results about convex functions to obtain some bounds. All the necessary results on convex functions are contained in the appendix or the exercises.

We shall limit our investigation to a particular type of extremal problem whose initiation is generally credited to Turán [19]. In this problem, we ask the following: Given a graph G, determine the maximum number of edges, $ex(n;\ G)$ in a graph of order n that does not contain G as a subgraph. A graph E of order n with $ex(n;\ G)$ edges and not containing G as a subgraph is called an *extremal graph* for this problem. The complete solution of any such problem ordinarily requires two things. First, we must produce an extremal graph on n vertices and $ex(n;\ G)$ edges that does not contain any G. Second, we must show that any graph on n vertices and with at least $ex(n;G) + 1$ edges must contain a G.

The investigation of this extremal problem will eventually lead us to the study of the structure of extremal graphs. A rather beautiful theory has been developed that essentially tells us that the exact structure of the forbidden graphs themselves is not really as important as their chromatic number.

Section 10.1 Complete Subgraphs

We begin with Turán's original problem: What is the maximum number of edges q in a graph of order n that does not contain the complete graph K_p? We begin by producing the extremal graph for the Turán problem. This graph is easy to describe. For the forbidden graph K_{p+1} (with chromatic number $p + 1$), we begin with the complete p-partite graph $K_{n_1, n_2, \ldots, n_p}$ where $n = \sum n_i$. It is easy to show that among all such graphs, the one with the maximum number of edges is that graph with partite sets as nearly equal as possible. In fact, among all graphs on n vertices with chromatic number p, it has the maximum number of edges. Thus, if $n = kp + r$, $0 \le r < p$, then $p - r$ of the partite sets contain k vertices and the remaining r of the partite sets contain $k + 1$ vertices. We denote this graph as $T_{n,p}$, and call it the *Turán graph.* We further note that

$$|E(T_{n,p})| = \binom{n}{2} - \frac{k(n - p + r)}{2}.$$

We are now ready to state Turán's theorem.

Theorem 10.1.1 Among the graphs of order n which do not contain K_p, there exists exactly one with the maximum number of edges, namely $T_{n,p-1}$.

We will present (or at least sketch) two proofs to Turán's theorem, showing two common and useful techniques in extremal theory. The first technique is called "chopping" and resembles Turán's original proof. The strategy is to chop off a "useful" subgraph and work around this structure to complete the proof, carefully avoiding the "chopped" graph. For convenience and to maintain a notation common in extremal theory, we denote by G^n a graph of order n.

Turán's Chopping Proof. We proceed by induction on n, the order of the extremal graph under construction. The anchor is trivial so assume the result holds for orders less than n and suppose the extremal graph G^n is $K_p -$ free. Since G is extremal, it follows that $H = K_{p-1} \subseteq G^n$ and define q_1, q_2, q_3 as follows:

$$q_1 = |E(H)| = \binom{p-1}{2},$$
$$q_2 = \text{no. of edges between } H \text{ and } V - H \le (n - p + 1)(p - 2)$$
$$q_3 = |E(V - H)| \le |E(T_{n-p+1,p-1})|$$

(the bound in the third expression follows from the inductive assumption).

It is clear that $|E(G^n)| = q_1 + q_2 + q_3$, and by summing the bounds given on q_1, q_2 and q_3 we see that

$$|E(G^n)| \le |E(T_{n,p-1})|.$$

It remains to show that if equality holds, then $G^n = T_{n,p-1}$. Clearly, $q_2 = (n - p + 1)(p - 2)$. This determines a partition of $V(G^n)$ into $p - 1$ classes, defined according to their $p - 2$ adjacencies in H. These classes are clearly independent, so G^n is a complete $(p - 1)$ partite graph defined by these classes, that is, $G = T_{n,\, p-1}$. □

The second proof technique, known as *symmetrization*, has become a powerful tool in extremal theory. The process of symmetrization proceeds as follows: Given nonadjacent vertices v and u, we delete all the edges incident to the vertex u and make u adjacent to all vertices in $N(v)$. The vertex u is then said to be *symmetric to v*. We can see that under certain conditions symmetrization can be useful in extremal problems. First, it is clear that no K_p is formed during this process, since only the vertices of $N(v)$ have new adjacencies and u and v are not adjacent. Thus, if a K_p now exists, it must have existed prior to symmetrization. Second, if $deg\ u < deg\ v$, then we have increased the number of edges in the graph without producing the forbidden K_p. We now sketch a second proof of Turán’s theorem using symmetrization from Zykov [21].

Sketch of Zykov’s Proof. We assume the anchor and inductive steps have been performed and consider the extremal graph G^n which is K_p-free. Let v have maximum degree in G^n and symmetrize all of $V - N(v)$ to v. Let S_1 denote these vertices along with v. Clearly, S_1 is an independent set of vertices. Further, since v had maximum degree, our new graph has at least as many edges as G^n. Now, repeat this process on $< G^n - S_1 >$, forming the set S_2. Continue the procedure, forming the sets $S_3, \ldots, S_d$. As we noted earlier, since G^n was K_p-free, this new graph formed by symmetrization is also K_p-free. (That is, $d \le p - 1$). Thus, any K_p-free graph can be transformed into a d-partite ($d \le p - 1$) graph. Further, to maximize the number of edges in such a graph, standard convexity arguments (as noted before) imply that the graph is actually $T_{n,p-1}$. □

The following is immediate from Turán’s Theorem, but was originally proven by Mantel in 1906 [13]. We provide an independent proof of this result because the technique is different and very interesting.

Corollary 10.1.1 If G^n is K_3-free, then $|E(G^n)| \le \dfrac{n^2}{4}$.

Proof: Suppose G is as described and number the vertices of G from 1 to n. Assign vertex i a weight of $w_i \ge 0$ such that $\sum_{i=1}^{n} w_i = 1$. Our goal is to maximize

$$S = \sum_{ij \in E(G)} w_i w_j$$

(where the sum is taken over all edges in G). Suppose vertices u and v are not adjacent in G. Let the neighbors of u have total weight x and let the neighbors of v have total weight y, where we assume without loss of generality that $x \geq y$.

Since

$$(w_u + \varepsilon)x + (w_v - \varepsilon)y \geq w_u x + w_v y$$

we do not decrease the value of S if we shift some weight from the vertex v to the vertex u. It follows that S is maximal if all the weight is concentrated on some complete subgraph of G, in fact, on one edge. But then $S \leq \frac{1}{4}$ (applying standard convexity). On the other hand, taking all $w_i = n^{-1}$, we see that $S \geq n^{-2}|E|$. But then these two inequalities imply that $|E| \leq \frac{n^2}{4}$. □

We now state an extension of Turán's theorem from Erdös [5] which can be used to provide yet another proof of Turán's theorem.

Theorem 10.1.2 Let G^n be a K_p-free graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Then there exists a $(p - 1)$ chromatic graph H^n which is K_p-free with degrees $s_1 \geq s_2 \geq \cdots \geq s_n$ and such that $s_i \geq d_i$ for every i.

We continue our investigation of complete subgraphs with a theorem from Dirac [4] that shows that we actually get more than a K_p once we have more than the extremal number of edges. To this end, we say that a graph H is *saturated* (in particular, we say a graph H is *G-saturated)* if H does not contain G and if the addition of any edge to H results in a graph that does contain G. This idea is similar to the technique we have often used in assuming maximal counterexamples. For example, the proof of Ore's theorem (Theorem 5.1.1) used this approach.

Theorem 10.1.3 If $n \geq r + 1$, then every $(n, ex(n, K_r) + 1)$−graph G contains a $K_{r+1} - e$.

Proof. We proceed by induction on n, the order of G. For $n = r + 1$, it is clear that having one more than the extremal number of edges forces $G = K_{r+1} - e$, and so we have the anchor step.

Now, we assume the result holds on all such graphs of order less than n and consider an $(n, ex(n; K_r) + 1)$ graph G. Let x have minimum degree $\delta(G)$. Then it is easily seen that $\delta(G) \leq \delta(T_{n,r})$, and so $|E(G - x)| \geq ex(n - 1; K_r) + 1$. Hence, by induction we see that $G - x$ contains $K_{r+1} - e$, and the result holds. □

For completeness, we now state the following corollary to Turán's theorem.

Corollary 10.1.2 (Zarankiewicz [20]) If G^n is K_r-free, then

$$\delta(G^n) \leq \left(1 - \frac{1}{r-1}\right)n = \frac{r-2}{r-1}n.$$

With considerable effort, one can improve upon the above corollary; however, we shall simply state this improvement.

Theorem 10.1.4 ([1]). If $\chi(G^n) \geq r$ and G^n is K_r-free, then

$$\delta(G^n) \leq \frac{(3r-7)}{(3r-4)}\, n.$$

We continue our investigation of complete subgraphs by counting triangles. Turán's theorem tells us when we can be sure one triangle exists, but our goal is to establish bounds on the number of triangles that exist in general. In first attacking this problem, we will find it useful to change the setting and sum the number of triangles that must be contained in a graph and its complement. Let $k_r(G)$ equal the number of K_r's contained in the graph G. Independent work of several people, including Goodman [11], Moon and Moser [14] and Lovász [12] all lead to the following result.

Theorem 10.1.5 Given an (n, q)–graph G with $(n, \overline{q})$ complement $\overline{G}$,

$$k_3(G) + k_3(\overline{G}) = \binom{n}{3} - (n-2)\, q + \sum_{i=1}^{n} \binom{deg\ v_i}{2}$$

$$= \binom{n}{3} - (n-2)\, \overline{q} + \sum_{i=1}^{n} \binom{n-1-deg\ v_i}{2}.$$

Proof. Consider the degree sequence of G. There are $\sum_{i=1}^{n} \binom{deg\ v_i}{2}$ pairs of adjacent edges of G and $\sum_{i=1}^{n} \binom{n-1-deg\ v_i}{2}$ pairs of adjacent edges in $\overline{G}$. The sum of these two numbers can be counted in another way as well. Each of the triangles in G and $\overline{G}$ contains three pairs of adjacent edges, and each of the remaining

$$L = \binom{n}{3} - k_3(G) - k_3(\overline{G})$$

triples of vertices contains exactly one such pair. Hence,

$$\sum_{i=1}^{n} \binom{deg\ v_i}{2} + \sum_{i=1}^{n} \binom{n-1-deg\ v_i}{2} = 3k_3(G) + 3k_3(\overline{G}) + L.$$

Solving for our desired sum yields,

$$k_3(G) + k_3(\overline{G}) = \frac{1}{2}\left[\sum_{i=1}^{n} \binom{deg\ v_i}{2} + \sum_{i=1}^{n} \binom{n-1-deg\ v_i}{2} - \binom{n}{3}\right].$$

But note that

$$\sum_{i=1}^{n} \binom{n-1-deg\ v_i}{2} = \sum_{i=1}^{n} \frac{(n-1-deg\ v_i)(n-2-deg\ v_i)}{2}$$

$$= \sum_{i=1}^{n} \left[\binom{n-1}{2} - (n-2)\,deg\ v_i + \binom{deg\ v_i}{2} \right].$$

Now, substituting and rearranging terms completes the result. □

Corollary 10.1.3 The graphs G^n and $\overline{G}^n$ contain a total of at least

$$\frac{n(n-1)(n-5)}{24} \text{ triangles.}$$

Theorem 10.1.6 An (n, q)–graph contains at least $\frac{q}{3n}(4q - n^2)$ triangles.

Proof. Suppose that $uv \in E$. Then there are at least $deg\ u + deg\ v - n$ vertices adjacent to both u and v. Thus, we see that

$$k_3(G) \geq \frac{1}{3} \sum_{uv \in E} (deg\ u + deg\ v - n).$$

But since each $deg\ u$ term appears $deg\ u$ times in this sum, we have that

$$k_3(G) \geq \frac{1}{3} \sum_{u \in V} deg^2\ u - nq.$$

So by the Cauchy inequality (see the appendix),

$$k_3(G) \geq \frac{1}{3}\left\{ \frac{(2q)^2}{n} - nq \right\} = \frac{q}{3n}[4q - n^2]. \quad \square$$

The next result is due to Rademacher (see [9]) and extends Mantel's Theorem.

Theorem 10.1.7 For every even n, a graph on n vertices with $\frac{n^2}{4} + 1$ edges contains at least $\frac{n}{2}$ triangles. Furthermore, this result is best possible.

The graph $K_{n/2,n/2} + e$ shows that Rademacher's Theorem is best possible. Our next result was originally conjectured by Nordhaus and Stewart [15]. The result is due to Bollobás [2].

Theorem 10.1.8 If G is a graph on n vertices and $\frac{n^2}{4} \leq |E(G)| \leq \frac{n^2}{3}$ edges then G contains at least $\frac{n}{9}(4|E(G)| - n^2)$ triangles.

Comparing these theorems over the range of possible values for q we see that Rademacher's Theorem is most acurate for $q = \frac{n^2}{4} + 1$ edges; the bounds of Moon and Moser and of Bollobás are equal when $q = \frac{n^2}{3}$; finally, Moon and Moser's Theorem yields the exact number of triangles when $q = \binom{n}{2}$.

We complete this section with a result due to Erdös [8]. We begin with a sequence of lemmas.

Lemma 10.1.1 Every $(n, ex(n-1, K_3) + 2)$−graph G which contains an odd cycle, contains a triangle.

Proof. Let G be as described and let C: $u_1, u_2, \ldots, u_{2k+1}$ be the vertices of a shortest odd cycle in G. We can assume that $3 < 2k + 1 \leq n$. Now $< u_1, u_2, \ldots, u_{2k+1} >$ can have no other edges, for otherwise a shorter odd cycle would be formed. Let $v_1, v_2, \ldots, v_{n-2k-1}$ be the other vertices of G. Any v_i $(1 \leq i \leq n - 2k - 1)$ can be adjacent to at most two u_j $(1 \leq j \leq 2k + 1)$, for otherwise an odd cycle shorter than C would be formed. Finally, Turá n's Theorem implies $< v_1, \ldots, v_{n-2k-1} >$ can have at most $ex(n - 2k - 1, K_3)$ edges. Thus, the number of edges in G is at most

$$2k + 1 + 2(n - 2k - 1) + ex(n - 2k - 1, K_3) \leq ex(n - 1, K_3) + 1,$$

contradicting our assumptions. □

Lemma 10.1.2 There exists a constant $c_2 > 0$ such that every $(n, ex(n, K_3) + 1)$

graph G contains at least $\lfloor c_2 n \rfloor$ triangles having a common edge (u, v).

Proof. Let $T = \{ (u_i, v_i, w_i) \mid 1 \le i \le r \}$ be a maximal system of disjoint triangles in G. Thus, in $G - T$ the remaining $n - 3r$ vertices contain no triangles and therefore have at most $ex(n - 3r, K_3)$ edges.

Denote by $G(i)$ the graph obtained from G by deleting the first $i - 1$ triangles of T. Further, let $deg_i u_i$, $deg_i v_i$ and $deg_i w_i$ be the degrees of u_i, v_i and w_i in $G(i)$.

We now show that for some i $(1 \le i \le r)$ we must have

$$deg_i u_i + deg_i v_i + deg_i w_i > n(1 + 9c_2) - 3i, \tag{1}$$

for if this failed to hold for any i, then the number of edges in G would be at most

$$\sum_{i=1}^{r} [\, n(1 + 9c_2) - 3i \,] + ex(n - 3r, K_3) < ex(n, K_3)$$

by a simple calculation for sufficiently small c_2. But this contradicts the fact G contains at least $ex(n, K_3) + 1$ edges. Thus, (1) holds for say $i = i_0$. Then a simple calculation shows that there are at least $3\lfloor c_2 n \rfloor$ vertices of $G(i_0)$ which are adjacent to more than one of the vertices $u_{i_0}, v_{i_0}, w_{i_0}$. Therefore, there are at least $\lfloor c_2 n \rfloor$ vertices adjacent to the same pair, which completes our proof. □

Lemma 10.1.3 Let $\delta > 0$ be a fixed number. Consider any (n, q)-graph G with $q > ex(n, K_3) - \frac{n}{2}(1 - \delta)$, $n > n_0(\delta)$, which contains a triangle. Then G contains an edge (u, v) and $r = \lfloor c_3 n \rfloor + 1$ $(c_3 = c_3(\delta))$ vertices w_i $(i = 1, 2, \ldots, r)$ so that all the triangles (u, v, w_i) $(i = 1, 2, \ldots, r)$ are in G.

Proof. By assumption, G contains a triangle (u, v, w). Assume first that

$$deg\ u + deg\ v + deg\ w > n(1 + 9c_3) + 9. \tag{2}$$

Then the result follows from Lemma 2.

If (2) fails to hold, then $G - u - v - w$ has $n - 3$ vertices and at least $q - n(1 + 9c_3) - 9$ edges. But if $c_3 < \frac{\delta}{18}$, then for $n > n_0$,

$$\begin{aligned} &q - n(1 + 9c_3) - 9 > ex(n, K_3) - \frac{n}{2}(1 - \delta) - n(1 + 9c_3) - 9 \\ &> ex(n - 3, K_3). \end{aligned}$$

But then by Lemma 10.1.2, $G - u - v - w$ contains the desired configuration of triangles, which completes the proof. □

We are finally ready to present our goal, a theorem due to Erdös [8].

Theorem 10.1.9 There exists a constant $c_1 > 0$ such that for n sufficiently large and $t < c_1 n/2$, if a graph G on n vertices contains at least $\lfloor \frac{n^2}{4} \rfloor + t$ edges, then G contains at least $t \lfloor \frac{n}{2} \rfloor$ triangles.

Proof. Suppose G is as above and $t < c_1 \frac{n}{2}$. We first assume that after the omission of any $r = \lfloor c_1 n/2c_3 \rfloor$ edges, the graph still contains a triangle. (Note: $c_3 = c_3(\delta)$, for $\delta = \frac{1}{4}$ in the last lemma.) For sufficiently small c_1, $\frac{c_1}{2c_3} < \frac{1}{4}$; thus it will be permissible to apply Lemma 10.1.3 during the omission of these edges.

By Lemma 10.1.3 (or Lemma 10.1.2) there exists an edge e_1 contained in $\lfloor c_3 n \rfloor + 1$ triangles of G. Again by Lemma 10.1.3 in $H_1 = G - e_1$, there exists an edge e_2 contained in at least $\lfloor c_3 n \rfloor + 1$ triangles of H_1. Suppose we have already chosen the edges $e_1, \ldots, e_r$ each of which is contained in at least $\lfloor c_3 n \rfloor + 1$ triangles. By our earlier assumption $H_r = G - e_1 - \cdots - e_r$ contains at least one triangle. But then by Lemma 10.1.3 there is an edge e_{r+1} in H_r which is contained in at least $\lfloor c_3 n \rfloor + 1$ triangles of H_r. These triangles incident on the edges $e_1, \ldots, e_{r+1}$ are clearly distinct, thus G contains at least

$$(r + 1) \{\lfloor c_3 n \rfloor + 1\} > c_1 \frac{n^2}{2} > t\frac{n}{2}$$

triangles, which completes the proof in this case.

Therefore, we may assume that there are $s \le r < \frac{n}{4}$ edges $e_1, e_2, \ldots, e_s$ so that the graph $H = G - e_1 - e_2 - \cdots - e_s$ contains no triangles and we may assume s is the smallest integer with this property. By the fact that $s \le r < \frac{n}{4}$, H has

$$ex(n, K_3) + t - s > ex(n, K_3) - \frac{n}{4} > ex(n - 1, K_3) + 1$$

edges. Thus, by Lemma 1, H must contain only even cycles.

By Theorem 10.1.1, $s \ge t$. Suppose $s = t$. Then H has $ex(n, K_3)$ edges and by Theorem 10.1.1, $H = T_{n,2}$. Clearly, the addition of any edge creates at least $\lfloor \frac{n}{2} \rfloor$ distinct triangles. A simple argument shows that the addition of every further edge introduces at least $\lfloor \frac{n}{2} \rfloor$ triangles and that these triangles are distinct. Thus, G contains at least $t \lfloor \frac{n}{2} \rfloor$ triangles and our result is shown in this case as well.

Finally assume $s = t + w$, $0 < w < \frac{n}{4}$ (since $s < n/4$). We also assume n is even, say $n = 2m$. Now since H contains only even cycles, it is a subgraph of a bipartite graph B whose vertices are say $\alpha_1, \ldots, \alpha_{m-u}$ and $\beta_1, \ldots, \beta_{m+u}$ (since H has more than $ex(2m, K_3) - \frac{m}{2}$ edges, we have $0 \le u < (m/2)^{1/2}$).

Clearly, every one of the edges $e_1, \ldots, e_s$ join two of the α's or two of the β's, for otherwise for some e_i, the graph $G - e_1 - \cdots - e_{i-1} - e_{i+1} - \cdots - e_s$ would still have only even cycles and hence no triangles, which contradicts the minimum property of s.

By our assumption, H is a subgraph of B. Assume H is obtained from B by the omission of x edges. Then we clearly have

$$s = x + u^2 + t \quad (\text{or} \quad w = x + u^2),$$

and G is obtained from H by adding s edges $e_1, \ldots, e_s$ which are all of the form $(\alpha_{i_1}, \alpha_{i_2})$ or $(\beta_{i_1}, \beta_{i_2})$. Let $e_i = (\beta_{i_1}, \beta_{i_2})$ and let us estimate the number of triangles $(\beta_{i_1}, \beta_{i_2}, \alpha_j)$ in B. Clearly, at most x of the edges (β_{i_1}, α_j), (β_{i_2}, α_j) are not in B; thus $B + e_i$ contains at least $m - u - x$ triangles (if e_i connects two α's, then $B + e_i$ contains at least $m + u - x$ triangles). For different e_i's these triangles are clearly different; thus $G = H + e_1 + \cdots + e_s$ contains at least

$$(m - u - x)s = (m - u - x)(x + u^2 + t) \ge tm = t(n/2)$$

triangles. The above follows by simple computation from $s = u^2 + x + t < m/2$. The above equation completes the proof in the $n = 2m$ case. For $n = 2m + 1$ the proof is almost identical and hence we omit it here. This completes our proof. □

This result has been improved by Lovász and Simonovits (see [3]) who showed that the theorem holds for $c_1 = 1$.

Section 10.2 Cycles in Graphs

We now modify the forbidden subgraph in question and consider extremal results involving cycles of various sizes. Our first result is somewhat less specific than those we have seen thus far. Our concern is in finding a pair of vertex disjoint cycles of unspecified order. We define $s(n)$ to be the minimum number of edges so that every graph on n vertices contains two vertex disjoint cycles. The following result of Pósa [16] was presented in Chapter 5. We restate it here for completeness.

Theorem 10.2.1 For $n \ge 6$, $s(n) = 3n - 5$.

Despite finding $s(n)$ precisely, we find the previous result somewhat unsatisfying in that we do not have any information about the size of the cycles that must exist. We now turn to a line of investigation for cycles similar to the one taken for complete graphs. We begin with a bound on the size that will ensure that a graph contains a 4-cycle. Our strategy will be to count pairs of vertices that are dominated by a common vertex. Allowing only one such neighbor will prevent 4-cycles.

Theorem 10.2.2 Every graph on n vertices and $q > \frac{n}{4}(1 + \sqrt{4n-3})$ edges contains a 4-cycle.

Proof. Suppose G^n contains no 4-cycle. Our strategy will be to count pairs of vertices that are dominated by another vertex and in doing so, to create an upper bound on the number of edges in G^n.

For a fixed $z \in V$, there are $\binom{deg\ z}{2}$ pairs of vertices dominated by z. On the other hand, each pair x, y is counted at most once, since if it were counted twice, a 4-cycle would have to exist. Hence,

$$\sum_{z \in V} \binom{deg\ z}{2} \leq \binom{n}{2}.$$

Now, by Jensen's Inequality (see the appendix),

$$\binom{n}{2} \geq \sum_{z \in V} \binom{deg\ z}{2} \geq n \binom{2qn^{-1}}{2} = \frac{2q^2 - nq}{n}.$$

Thus,

$$q^2 - \frac{nq}{2} \leq \frac{n^3 - n^2}{4}$$

and hence,

$$q \leq \left(\frac{(4n^3 - 3n^2)}{16} \right)^{½} + \frac{n}{4}$$
$$= \left(\frac{n}{4}\right)(1 + \sqrt{4n - 3}).$$

Thus, the result follows. We note that we often say that the extremal value for C_4 is $O(n^{3/2})$; that is, q is bounded by a function that is on the order of $O(n^{3/2})$. □

In attempting to produce a sharp extremal example for the previous result, we come across our first difficulty. However, with the use of projective planes, we can produce a meaningful example. A *finite projective plane* is a special type of block design. That is, it is a family of v points and b subsets of these points such that each subset contains k of the points and each pair of points occur together in exactly λ of the subsets. For finite projective planes, a more geometric view is taken and the parameters carry special restrictions. First of all, $k = n + 1$ and $\lambda = 1$. The sets are called *lines,* and the following four properties (defining properties of block designs) hold.

1. Any line is incident with $n + 1$ points.
2. Any point is incident with $n + 1$ lines.
3. Any pair of points are joined by exactly one line.
4. Any pair of lines intersect in exactly one point.

We can model projective planes as bipartite graphs as follows. Define the *point-line incidence graph* G of a projective plane of order p, where p is a prime as follows: The vertices of G correspond to the points and lines of the projective plane. A line is joined to each of the points that lie on the line. Thus, the point-line incidence graph is an $n + 1$ regular bipartite graph of order $2x = 2(n^2 + n + 1)$. Further, it can be shown (in the exercises) that this graph has no cycle smaller than a 6-cycle. In fact, it represents the $(n + 1)$-regular graphs of smallest order having this property. Further,

$$|E(G)| = x(n + 1) = \frac{x}{2}(1 + \sqrt{4x - 3}), \text{ where } x = \frac{|V(G)|}{2}.$$

For $n = 2$, the projective plane can be modeled as shown in Figure 10.2.1. Can you construct the point-line incidence graph for the projective plane of order 2?

We can find a natural setting in which the above examples do provide a set of extremal graphs. Suppose we consider a variation on the usual extremal question in which we consider only bipartite graphs rather than general graphs. It is natural to forbid only other bipartite graphs and, in particular, complete bipartite graphs. With this in mind, we define $ex(n, m; K_{s,t})$ to be the maximum number of edges in a bipartite graph with partite sets of orders $n \geq s$ and $m \geq t$ that does not contain a $K_{s,t}$. In particular, then, we first investigate forbidden C_4s; that is, we seek the extremal value $ex(n, n; K_{2,2})$.

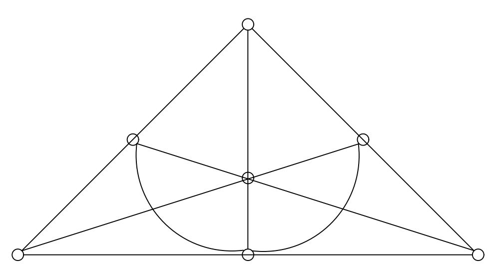

Figure 10.2.1. A model of the projective plane of order $p = 2$.

Theorem 10.2.3 For $n \geq 2$, $ex(n, n; K_{2,2}) \leq \frac{n}{2}(1 + \sqrt{4n - 3})$, and equality holds infinitely often.

Proof. Let $q = \frac{n}{2}(1 + \sqrt{4n - 3})$ and note that

$$(q - n)q = n^2(n - 1).$$

Now, suppose there is a bipartite graph $G_{n,n}$ of size greater than q that does not contain a $K_{2,2}$. Denote by $d_1, d_2, \ldots, d_n$ the degrees of the vertices in the first partite set V_1. Then,

$$\sum_{i=1}^{n} d_i = |E(G_{n,n})| = e > q$$

and

$$\binom{n}{2} \geq \sum_{i=1}^{n} \binom{d_i}{2} = \frac{1}{2}\sum_{i=1}^{n} d_i^2 - \frac{1}{2}\sum_{i=1}^{n} d_i$$

$$\geq \frac{1}{2n}e^2 - \frac{e}{2} > \frac{q(q-n)}{2n} = \binom{n}{2}.$$

Thus, we reach a contradiction and the bound now follows.

The previous example of the point-line incidence graph of the projective plane shows that equality holds infinitely often . □

We now consider the more general bipartite extremal problem. The following lemma will be useful in our computations. This approach follows Bollobás [3].

Lemma 10.2.1 Let m, n, s, t, r, k be integers with $2 \leq s \leq m$, $2 \leq t \leq n$, $k \geq 0$, $0 \leq r < m$ and $G_{m,n}$ of size $mx = km + r$ and with no $K_{s,t}$; then

$$m\binom{x}{t} \leq (m - r)\binom{k}{t} + r\binom{k+1}{t} \leq (s-1)\binom{n}{t}.$$

Proof. Let $T \subseteq V_2$ with $|T| = t$ and say T is linked to $v \in V_1$ if $T \subseteq N(v)$. Then the number of t-sets linked to v is $\binom{deg\ v}{t}$. Since by assumption $G_{m,n}$ contains no $K_{s,t}$, each t-set in V_2 is linked to at most $s - 1$ vertices of V_1. Hence,

$$m\binom{x}{t} \le \sum_{v\in V_1}\binom{deg\ v}{t} \le (s-1)\binom{n}{t}.$$

Since $\sum_{v\in V_1} deg\ v = mx = km + r,\ 0 \le r < m$, and since $f(w) = \binom{w}{t}$ is convex, the inequality follows. □

Theorem 10.2.4 For $t \le s$,

$$ex(m,\ n;\ K_{s,\ t}) \le (s-1)^{1/t}\ (n-t+1)m^{1-1/t} + (t-1)m.$$

Proof. Let $G_{m,\ n}$ be an extremal bipartite graph with $ex(m,\ n;\ K_{s,t}) = mx$ edges which is $K_{s,t}$-free. As $x < n$, the lemma implies that

$$(x-(t-1))^t \le (s-1)(n-(t-1))^t m^{-1}.$$

Thus,

$$x-(t-1) \le (s-1)^{1/t}\ (n-t+1)\ m^{-1/t},$$

or

$$mx \le (s-1)^{1/t}\ (n-t+1)\ m^{1-1/t} + (t-1)m. \quad \square$$

Corollary 10.2.1 If $t \ge 2$ and c is a constant such that $c > (t-1)^{1/t}$ (t, c fixed), then for n sufficiently large,

$$ex(n,\ n;\ K_{t,t}) < cn^{2-1/t}.$$

We can use the bound in Theorem 10.2.4 to obtain a bound in general graphs.

Theorem 10.2.5 Let G be an $(n,\ q)$-graph that does not contain a $K_{s,t}$, $2 \le s,\ \ 2 \le t$. Then

$$q \le \frac{1}{2}\ ex(n,\ n\ ;\ K_{s,t})$$

Proof. We construct a bipartite graph B from G as follows. Let V_1 and V_2 be copies of $V(G)$. For every edge xy in G, insert the edge from $x_1 \in V_1$ to $y_2 \in V_2$ (hence, two edges are placed into B). Then B has $2q$ edges and does not contain a $K_{s,t}$ and, hence, the result follows. □

Section 10.3 On the Structure of Extremal Graphs

In this section we begin an investigation of the structure of extremal graphs. After determining the extremal values of various forbidden graphs, it is natural to try to gain further information about the extremal graphs themselves. It is not surprising that a great deal can be said and that this information opens still other avenues of investigation. It should be noted that there is a fundamental difference between extremal problems in which one of the forbidden graphs is bipartite (called a *degenerate extremal problem*), and one where none of the graphs is bipartite. The reasons for this will become more apparent as we progress. For now, simply note that in the degenerate case $ex(n, H) = o(n^2)$, while in the nondegenerate case,

$$ex(n, H) \geq \left\lfloor \frac{n^2}{4} \right\rfloor.$$

The foundation for this section is primarily the work of Erdös and Stone [10]. Our goal is to show that for a class of graphs *H*, the extremal number $ex(n; H)$ depends only loosely on the graphs in *H*. That is, the exact structure of the forbidden subgraphs is not the critical issue, but rather the dominant feature is the minimum chromatic number of a graph in the class *H*. In what follows we use the notation $K_{(s)(t)}$ to mean the complete *s*-partite graph with *t* vertices in each partite set. We begin with two beautiful results from Erdös and Stone [10].

Theorem 10.3.1 Let $\varepsilon > 0$ and $k, t \geq 1$ be given. Then, for *n* sufficiently large, every graph of order *n* and with $\delta \geq (1 - \frac{1}{k} + \varepsilon)n$ contains $K_{(k+1)(t)}$.

Proof. (By induction on *k*). For $k = 1$, the statement claims that $\delta \geq \varepsilon n$ and, hence, *G* has at least $\frac{\varepsilon}{2}n^2$ edges. That *G* contains a $K_{t,t}$ follows from Corollary 10.2.1 and Theorem 10.2.5.

Now let $k \geq 2$ and $s = \lfloor \frac{1}{\varepsilon} t \rfloor$. If *n* is sufficiently large, then by our induction assumption, we can find a $K_{(k)(s)}$ in *G*. Let $Y = V(G) - K_{(k)(s)}$ and let *X* be those vertices of *Y* that are adjacent to at least *t* vertices in each of the partite sets of the $K_{(k)(s)}$. Then the number of missing edges between $Y - X$ and $K_{(k)(s)}$ is at least

$$\begin{aligned}(|Y| - |X|)(s - t) &\geq (|Y| - |X|)(1 - \varepsilon)s \\ &= (n - ks - |X|)(1 - \varepsilon)s.\end{aligned}$$

Also, the number of edges missing from any vertex in $K_{(k)(s)}$ is at most $(\frac{1}{k} - \varepsilon)n$. Thus, the number of edges missing from the vertices in $K_{(k)(s)}$ is at most

$$ks(\frac{1}{k} - \varepsilon)n = (1 - k\,\varepsilon)sn.$$

Thus, the preceding two inequalities imply that

$$(n - ks - |X|)(1 - \varepsilon)s \le (1 - k\,\varepsilon)sn,$$

and solving we see that $|X| \ge \frac{\varepsilon(k-1)}{(1-\varepsilon)}\ n - ks.$

Since $k \ge 2$ and $\varepsilon > 0$, we see that $|X|$ grows large as n grows large. Then, if

$$|X| > \binom{s}{t}^k (t - 1)$$

we can select t vertices that will form the final partite set we desire. □

Theorem 10.3.2 Let G be a graph of order n with at least $(1 - \frac{1}{k} + \varepsilon)\ \frac{n^2}{2}$ edges. Then for n sufficiently large, G contains a $K_{(k+1)(t)}$.

Proof. Remove a vertex of degree less than $(1 - \frac{1}{k} + \frac{\varepsilon}{2})\ |V(G)|$ if any exist. Now, in the graph that remains, repeat this process and continue to repeat this process as often as possible. Suppose that at some point in this process we are unable to continue; that is, suppose we are left with a graph H in which all vertices have degree at least $(1 - \frac{1}{k} + \frac{\varepsilon}{2})\ |V(H)|$. Let $|V(H)| = N$; then if N is sufficiently large, the result will follow from our last theorem. Then, all that remains is for us to show that N cannot be "too small." That is, we wish to show that N is bounded below by a function that grows as n grows.

In the construction of H, the number of edges we removed is at most

$$\sum_{j=N+1}^{n} j(1 - \frac{1}{k} + \frac{\varepsilon}{2}) = \left(\binom{n+1}{2} - \binom{N+1}{2}\right)(1 - \frac{1}{k} + \frac{\varepsilon}{2})$$
$$\le \left(\binom{n}{2} - \binom{N}{2}\right)(1 - \frac{1}{k} + \frac{\varepsilon}{2}) + (n - N).$$

The graph H has at most $\binom{N}{2}$ edges, and, thus,

$$(1 - \frac{1}{k} + \varepsilon) \binom{n}{2} \leq | E(G) |$$
$$\leq (1 - \frac{1}{k} + \frac{\varepsilon}{2}) [\binom{n}{2} - \binom{N}{2}]$$
$$+ (n - N) + \binom{N}{2}.$$

Thus,

$$\frac{\varepsilon}{2} \binom{n}{2} \leq (\frac{1}{k} - \frac{\varepsilon}{2}) \binom{N}{2} + (n - N).$$

Hence, we see that N grows large if n grows large.

Finally, to see that the process of removing vertices of small degree must stop, suppose that it does not stop and examine the sum on the number of edges removed (as we did above). In this case we would have at most $(1 - \frac{1}{k} + \frac{\varepsilon}{2}) \frac{n^2}{2}$ edges in G, a contradiction. Hence, the process must stop and the result is proved . □

The next, somewhat surprising result has been the goal of our work in this section. It tells us that the forbidden subgraph's structure is only somewhat responsible for the extremal number. That is, the exact structure of the graph is not as important as the chromatic number. The significance of the next result has lead to the following definition: Given a family of graphs F, the *subchromatic number* is defined to be

$$\Psi(F) = \min \{ \chi(G) : G \in F \} - 1.$$

The following result of Erdös and Simonovits [9] is an easy consequence of the Erdös - Stone Theorems.

Theorem 10.3.3 If F is a family of graphs with $\psi(F) = p$, then

$$ex(n, F) = (1 - \frac{1}{p}) \binom{n}{2} + o(n^2).$$

Proof. Since each $G \in F$ is not p-colorable, we see that G is not a subgraph of $T_{n,p}$. Hence,

$$ex(n, F) \geq |E(T_{n,p})| = (1 - \frac{1}{p}) \frac{n^2}{2} + O(n).$$

On the other hand, there is some $G_0 \in F$ with $\chi(G_0) = p + 1$ and say $|V(G_0)| = m$. Now the Erdös - Stone Theorem (10.3.2) asserts that

$$ex(n,\ K_{(p+1)(m)}) = (1 - \frac{1}{p}) \binom{n}{2} + o(n^2).$$

Since G_0 is a subgraph of $K_{(p+1)(m)}$, we have that

$$ex(n,\ F) \le ex(n,\ K_{(p+1)(m)}) \le (1 - \frac{1}{p} + o(1)) \binom{n}{2}.\square$$

The following is an immediate corollary.

Corollary 10.3.1

$$\lim_{n\to\infty} \frac{ex(n;\ G)}{n^2} = \frac{1}{2} (1 - \frac{1}{\chi(G) - 1}).$$

The structure of extremal graphs is fairly stable, in the sense that graphs that are nearly extremal (that is, do not contain the forbidden graph or graphs but have nearly as many edges as the extremal graphs) have a structure that is close to that of extremal graphs. That is, we need not make a great many changes in the edge set of a nearly extremal graph to obtain an extremal graph. This idea is expressed in our next result, the combined efforts of Erdös [6], [7] and Simonovits [18].

Theorem 10.3.5 (The First Stability Theorem) Let F be a family of forbidden graphs with subchromatic number p. For every $\varepsilon > 0$, there exists a $\delta > 0$ and an n_ε such that if G^n is F-free and if, for $n > n_\varepsilon$,

$$|E(G^n)| > ex(n;\ F) - \delta n^2,$$

then G^n can be obtained from $T_{n,p}$ by changing at most εn^2 edges.

The name "first stability theorem" clearly implies that there are others. Unfortunately, these results are beyond the scope of this text, but the interested reader is advised to see [17] and [3]. Our next theorem can be proven using the first stability theorem (10.3.5) and is due to the combined work of Erdös and Simonovits [6], [7] and [18].

For our next result we need the following idea. Consider a partition of the vertex set of G^n as say $S_1, \ldots, S_p$ and the p-partite graph $K_{s_1, \ldots, s_p}$ corresponding to this partition of $V(G^n)$, where $s_i = |S_i|$. An edge vw is called an *extra edge* if it is not in $K_{s_1, \ldots, s_p}$ but is in G^n (similarly, an edge is missing if it is in $K_{s_1, \ldots, s_p}$ but not in

G^n). For a given p, the partition $S_1, \ldots, S_p$ is *optimal* if the number of missing edges is minimum. Finally, for a given vertex v, let $b(v)$ denote the number of extra edges at v.

Theorem 10.3.6 (The Asymptotic Structure Theorem) Let F be a family of forbidden subgraphs with $\Psi(F) = p$. If S^n is any extremal graph for F, then it can be obtained from $T_{n,p}$ by deleting and adding at most $o(n^2)$ edges. Furthermore, if F is a finite family, then

$$\frac{\delta(S^n)}{n} = 1 - \frac{1}{p} + o(1).$$

Sketch of Proof. The first part of the theorem follows from Theorems 10.3.3 and the First Stability Theorem.

For the second part, consider an optimal partition $R_1, R_2, \ldots, R_p$ of $V(S^n)$ and assume R_1 has minimum order. Then, $| R_1 | \le \frac{n}{p}$ and by the First Stability Theorem

$$\sum_{v \in R_1} b(v) = o(n^2).$$

If r denotes the maximum order of a graph in F, take r vertices $v_1, \ldots, v_r$ with $\sum_{i=1}^{r} b(v_i)$ minimum. Clearly for some $c > 0$, $| R_1 | > cn$. Thus,

$$\sum_{i=1}^{r} b(v_i) \le \frac{r}{|R_1|} \sum_{v \in R_1} b(v) = o(n).$$

Now apply symmetrization in a slightly modified form. For an arbitrary vertex v in S^n, delete all incident edges and join v to all vertices adjacent to each of $v_1, \ldots, v_r$. The resulting graph S^* contains no member of F. Further, $| E(S^n) | \le | E(S^*) |$. Hence,

$$deg\ v = \left| \bigcap_{i=1}^{r} N(v_i) \right| \ge \left| \bigcup_{j=2}^{r} R_j \right| - \sum_{i=1}^{r} b(v_i) \ge n - \frac{n}{p} - o(n)$$

and the result follows. □

We next present an easy but useful result on the behavior of $ex(n; F)$.

Theorem 10.3.7 For every family F, $\dfrac{ex(n;\ F)}{\binom{n}{2}}$ is decreasing as $n \to \infty$.

Proof. For a fixed extremal graph H^m, take all $\binom{m}{n}$ subgraphs of order n, say

$G_1, \ldots, G_t$. Each edge of H^m is in $\binom{m-2}{n-2}$ of the G_is and, thus,

$$\binom{m-2}{n-2} |E(H^m)| \le \sum_{i<t} |E(G_i)| \le \binom{m}{n} |E(H^n)|.$$

But this implies that

$$\frac{|E(H^m)|}{\binom{m}{2}} \le \frac{|E(H^n)|}{\binom{n}{2}}. \quad \square$$

We finish our study of the structure of extremal graphs by trying to determine when the Turán graph is the extremal graph for a family of graphs *F*. We will see that $T_{n,p}$ is fundamental to extremal graphs.

To prove the next result, we use the technique of progressive induction. Essentially, the technique is as follows. For a given problem you are able to prove the inductive step under the assumptions of the inductive hypothesis. However, you are unable to prove the anchor step. (This could be because the anchor step is not true for small values.) It also appears that the proof of the anchor step is as difficult as a direct proof of the result. Thus, to establish the result, we define a function, say *D*, used to measure the "distance between our knowledge and the conjecture." We then attempt to show that the value of this measure must approach zero. We now use progressive induction to establish our next result. Once again this result is due to Simonovits [18].

Theorem 10.3.4 A family *F* has $T_{n,p}$ as an extremal graph (for *n* sufficiently large) if, and only if, some $G \in F$ has an edge *e* such that $p = \chi(G - e) = \Psi(F)$. Furthermore, if $T_{n,p}$ is extremal for *F* for infinitely many values of *n*, then it is the only extremal graph (again, provided *n* is sufficiently large).

Proof. One direction is easy, for if $\chi(G - e) \ge p + 1$ for every graph $G \in F$ and for every edge *e* of *G*, then the addition of one edge to $T_{n,p}$ cannot produce a graph that contains one of the forbidden graphs of *F*. If it did, that graph would necessarily have chromatic number *p* after the deletion of some edge. Thus, $T_{n,p}$ is not extremal for the family *F*.

For the other direction, we assume that $\chi(G) = p + 1$ but that $\chi(G - e) = p$. Further suppose that $\{E^n\}$ is a sequence of extremal graphs for the family *F*. Since $T_{n,p}$ contains none of the forbidden subgraphs, $|E(E^n)| \ge |E(T_{n,p})|$. We now define our measure of the "distance between our knowledge and the conjecture." Let

$$D(n) = |E(E^n)| - |E(T_{n,p})|.$$

In order to accomplish our goal we shall prove the following statement.

(*) For $n > n_0$, either $T_{n,p}$ is the only extremal graph or there is an $n' < n$ such that $D(n') > D(n)$ and $n' \to \infty$ as $n \to \infty$.

The implications of statement (*) are that for $n > n_0$, $T_{n,p}$ is the only extremal graph for the family F. In fact, for $n < n_0$, $D(n)$ is bounded by some constant c. Using (*), then, $D(n) \leq c$ for every n.

If we define N_i such that if $n > N_i$, then $n' > N_{i-1}$ and if $N_0 = n_0$, it is then easy to show by induction on i that for $n > N_i$, either $D(n) = 0$ or $D(n) \leq c - i$. Hence, for $n_1 = N_{k+1}$ (since $D(n) \geq 0$) $T_{n,p}$ is the only extremal graph.

To prove (*), we choose an arbitrary sequence of extremal graphs $\{E^n\}$ and applying Theorem 10.3.2 and the fact that

$$|E(E^n)| \geq |E(T_{n,p})|,$$

we know that $T_{pt,p} \subseteq E^n$. Similarly, we know that $T_{pt,p} \subseteq T_{n,p}$.

Let $S = E^n - T_{pt,p}$ and $T = T_{n,p} - T_{pt,p}$. Also, denote by e_S the number of edges from $T_{pt,p}$ to S and e_T the number of edges between $T_{pt,p}$ and T. It is easily seen that $e_T = (n - pt)(p - 1)t$.

Since E^n is extremal, $T_{pt,p}$ is an induced subgraph of E^n and each vertex of S is adjacent to at most $(p - 1)t$ vertices of $T_{pt,p}$. Therefore,

$$|E(E^n)| = |E(T_{pt,p})| + e_s + |E(S)|$$

$$\leq |E(T_{pt,p})| + (n - pt)(p - 1)\ t + |E(S)|$$

Using the above we see that

$$D(n - pt) - D(n) = [|E(S)| - |E(T)|] - [|E(E^n)| - |E(T_{n,p})|]$$

$$= [|E(T_{n,p})| - |E(T)|] - [|E(E^n)| - |E(S)|]$$

$$\geq e_T - e_S$$

$$\geq 0.$$

The only thing left for us to do is to check that if $D(n) = D(n - pt)$, then $E^n = T_{n,p}$. Clearly, each vertex of S is joined to exactly $p - 1$ classes of $T_{pt,p}$. We partition the vertices of E^n into p sets $A_1, \ldots, A_p$ by placing in A_j all vertices of E^n that are not adjacent to vertices in the jth partite set of $T_{pt,p}$. The vertices of A_j are clearly independent; otherwise, some $H \in F$ must be in E^n. Thus, E^n must be a p-colorable graph. Recall that it has at least as many edges as $T_{n,p}$. Therefore, we have that $E^n = T_{n,p}$, and the proof is complete. □

Exercises

1. Complete the arithmetic on $q_1 + q_2 + q_3$ in the proof of Theorem 10.1.1 that establishes the bound on $|E(G^n)|$.
2. Prove Theorem 10.1.2.
3. Provide a graph theoretic proof of the fact that the maximum number of edges in a complete d-partite graph of order n ($d \leq p - 1$) actually occurs in the graph $T_{n,\, p-1}$ (see Zykov's proof).
4. Prove Mantel's Theorem directly, without using Turan's theorem or proof. (Hint: Use a counting argument initially similar to that of Theorem 10.1.5).
5. Verify Corollary 10.1.2.
6. If G is a k-regular graph of order n then

$$k_3(G) + k_3(\overline{G}) = \binom{n}{3} - \frac{nk(n-k-1)}{2}.$$

7. Prove Corollary 10.1.3.
8. Let G be a graph of order n and let N_k denote the number of K_ks in G. Prove that

$$\frac{N_{k+1}}{N_k} \geq \frac{1}{k^2-1}\left(k^2 \frac{N_k}{N_{k+1}} - n\right).$$

9. Let G be a graph on mk vertices and more than $\binom{k}{2} m^2$ edges. Prove that G contains a K_{k+1}.
10. (Stronger version of Theorem 10.3.2) Let $k \geq 2$ be an integer and let $0 < \varepsilon < ½(r-1)$. Then there exists a $d = d(\varepsilon, r) > 0$ such that if n is sufficiently large and

$$q > \left\{\frac{k-2}{k-1} + \varepsilon\right\}\frac{n^2}{2}$$

then every graph of order n and size q contains a $K_{k(t)}$ with $t \geq \lfloor d \log n \rfloor$. Hint: use induction, Theorem 10.2.4 with $s = 2$ and

$$\varepsilon_k = ½\left\{\frac{k-2}{k-1} - \frac{k-3}{k-2}\right\} = \{2(k-1)(k-2)\}^{-1} > 0.$$

11. If $|E(G)| = (1 - \frac{1}{r})\frac{n^2}{2}$, then $N_k \geq \binom{r}{k} \left(\frac{n}{k}\right)^k$, ($k \leq r + 1$, r real).
12. A strongly connected tournament on n vertices contains at least $\binom{n-1}{2}$ cycles.

13. A tournament of order n contains at least one and at most $n!/2^{\frac{n}{2}}$ hamiltonian paths.

14. Every tournament T of order n contains a transitive tournament of order at least $[\log_2 n] + 1$.

15. If T is a strong tournament, then the number of transitive subtournaments of order k $(k \geq 3)$ is at most

$$\binom{n}{k} - \binom{n-2}{k-2}.$$

16. Suppose G is a (p, q) graph with at most $q = |E(T_{r,p})|$ but G is not $T_{r,p}$. Then G contains a subgraph H of order $r + p$ and size

$$\binom{r+p}{2} - p + 1.$$

That is, there are less than p edges missing from H.

17. Show that if G is a graph of order p with

$$\delta(G) \geq \left\lfloor \frac{(r-2)p}{(r-1)} \right\rfloor + 1,$$

then G contains a K_r.

18. Suppose that G has order $p \geq r + 1$ and size $q = ex(p; K_r) + 1$.
(a) Show that G contains two copies of K_r with $r - 1$ vertices in common.
(b) Show that for every k, $r \leq k \leq p$, G has a subgraph with k vertices and at least $ex(k, K_{r-1}) + 1$ edges.

19. Show that every graph of order $p \geq 5$ and size $q \geq \left\lfloor \frac{p^2}{4} \right\rfloor + 2$ contains two triangles with exactly one vertex in common.

20. Let $0 < a < a + \varepsilon < 1$. Then every graph G of order $n \geq 2a/\varepsilon$ and size at least ½ $(a + \varepsilon)n^2$ contains a subgraph H with $|V(H)| = h \geq (\frac{\varepsilon}{2})n$ and minimum degree at least ah.

21. Let $r \geq 2$, $1 \leq t \leq q$ and $N = n - (r-1)q \geq 1$. Let G be a graph of order n that contains a $K_{(r-1)(q)}$, but does not contain a $K_{r(t)}$. Then G has at most $e = ((r-1)q + t)N + 2qN^{1-\frac{1}{t}}$ edges of the form xy, where $x \in K_{(r-1)(q)}$ and $y \in G - K_{(r-1)(q)}$.

22. Let $F = K_{r(t)}$, where $r \geq 2$ and $t \geq 1$. Then the maximal size of a graph of order n without F is

$$ex(n;\ F) = \tfrac{1}{2}\ \{\frac{(r-2)}{(r-1)} + o(1)\}n^2.$$

23. Let $F_1, F_2, \ldots, F_j$ be non-empty graphs. Denote by r the minimum of the chromatic numbers of the F_i. Then the maximal size of a graph of order n not containing any of the F_i is

$$ex(n;\ F_1, \ldots, F_j) = \tfrac{1}{2}\{\ \frac{(r-2)}{(r-1)} + o(1)\ \}n^2.$$

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