

A(1)

ANSWERS OR HINTS TO SELECTED PROBLEMS

From ~~Second~~^{Third} Edition of

"How To Prove It" by Daniel Velleman (2019/2006)

~~Third Edition Problems are in RED.~~

Ch. 1.1 #1(a) $(R \vee H) \wedge (\neg(H \wedge T))$

✓ $R =$ we'll have a reading assignment $H =$ we'll have homework problems $T =$ we'll have a test,

(c) $\neg P$ where $P = " \sqrt{7} \leq 2 "$

(b) $(\neg K) \vee (K \wedge \neg N)$ $K =$ skiing, $N =$ snow

2 (a) $(J \wedge B) \wedge (\neg J \wedge \neg B)$ $J =$ John is telling the truth

✓ (b) $(F \vee C) \wedge (\neg(F \wedge P))$ $F =$ fish, $C =$ chicken; $P =$ potatoes.

(c) $P \wedge Q \wedge R$; $P = 3/6$, $Q = 3/9$, $R = 3/15$

5

4. (a) and (c) are well-formed formulas

(b) and (d) are not

6

5. (a) It is not the case that I will buy the pants
and not buy the shirt

(b) I will not buy the pants and I will not buy the shirt

7 6. (a) Steve or George are happy, and Steve or George are unhappy

(b) Steve is happy, or George is happy and Steve is
unhappy, or George is unhappy

9

7 (a) $\neg(J_M \wedge P_M) \wedge (P_M \vee P_C) \wedge J_M \therefore P_C$ valid(b) $(M_B \vee M_F) \wedge (V_P \vee V_C) \wedge \neg(M_F \wedge V_C) \therefore \neg(M_B \wedge V_P)$ not valid(c) $(J \vee B) \wedge (\neg S \vee \neg B) \therefore (J \vee \neg S)$ valid(d) $(S \wedge H) \vee (E \wedge \neg H) \therefore \neg(S \wedge E)$ not valid

A(2)

Ch.1.2 #1 (a) $P \quad Q \quad \neg P \vee Q$

		$\neg P \vee Q$			$\neg [P \wedge (Q \vee \neg P)]$
T	T	T	T	T	F
T	F	F	T	F	T
F	T	T	F	T	T
F	F	T	F	F	T

1(b) $G \quad S \quad (S \vee G) \wedge (\neg S \vee \neg G)$

		$(S \vee G) \wedge (\neg S \vee \neg G)$			Have fun!
T	T	F			
T	F	T			
F	T	T			
F	F	F			

3(a) $P \quad Q \quad P \# Q$

		$P \# Q$			3(b) $P \quad Q \quad (P \vee Q) \wedge (\neg (P \wedge Q))$
T	T	F	T	T	F
T	F	F	T	F	T
F	T	T	F	T	T
F	F	F	F	T	F

4(a) $P \vee Q$ is equivalent to $\neg(\neg P \wedge \neg Q)$
 (b) Use a truth table to check this

5(a) $P \quad Q \quad P \downarrow Q$

		$P \downarrow Q$			5(b) $P \downarrow Q$ is equiv. to $\neg(P \vee Q)$
T	T	F			
T	F	F			
F	T	F			
F	F	T			

5(c) $P \downarrow P$ is equiv. to $\neg P$

$$(P \downarrow Q) \downarrow (P \downarrow Q) \quad " \quad P \vee Q$$

$$(P \downarrow P) \downarrow (Q \downarrow Q) \quad " \quad P \wedge Q$$

7. The argument will be valid if the conclusion has truth value T whenever all the premises have truth value T.

A(3)

- Ch. 1.2 #9 (a) neither
 ✓ (c) tautology (b) contradiction
 (d) tautology

15. 2^n

✓ 16. $P \vee (\neg Q)$

✓ 17. $[(\neg P) \wedge Q] \vee [P \wedge (\neg Q)]$

16. $(\neg P \wedge \neg Q) \vee (P \wedge \neg Q) \vee (P \wedge Q)$ (another answer)

- Ch. 1.3 #1 (a) $D(6,3) \wedge D(9,3) \wedge D(15,3)$
 (b) $D(x,2) \wedge D(x,3) \wedge \neg D(x,4)$
 (c) $(N(x) \wedge N(y)) \wedge (P(x) \vee P(y)) \wedge \neg (P(x) \wedge P(y))$
 2. (a) $(M(x) \wedge M(y)) \wedge (T(x,y) \vee T(y,x))$
 ✓ (b) $(B(x) \vee B(y)) \wedge (R(x) \vee R(y))$

- 3 (a) $\{x : x \text{ is a planet in our solar system}\}$
 ✓ (b) $\{x : x \text{ is an Ivy league school}\}$
 4. (a) $\{x : x \text{ is the square of a positive integer}\}$
 ✓ (b) $\{x : x \text{ is a non-neg. integer power of } 2\}$

5. (a) x is bound, no free variables, TRUE (b) x is bound, no free var., FALSE
 (c) x is bound, c is free
 6. (a) x is bound, w and c are free (b) x is bound, y is bound, no free var., TRUE
 (c) x and y are bound, no free variable, FALSE

8. (a) $\{x : x \text{ was married to Elizabeth Taylor}\} = \{\text{Richard Burton}, \dots\}$

- (b) $\{x : x \text{ is a logical connective studied in Section 1.1}\} =$

9. (a) $\{x : x \in \mathbb{R} \text{ and } x^2 - 4x + 3 = 0\} = \{1, 3\} = \{\top, \wedge, \vee\}$
 (b) $\{x : x \in \mathbb{R} \text{ and } x^2 - 2x + 3 = 0\} = \emptyset$
 (c) $\{x : x^2 + 5^2 < 50\} = \{x : x^2 < 25\} = (-5, 5)$

8. (c) $\{x : x = \text{Daniel J. Velleman}\} = \{\text{Daniel J. Velleman}\}$

A(4)

Ch. 1.4 #1 (a) $A \cap B = \{3, 12\}$

(d) $A - B = \{1, 35\}$

 $A \cap B$ and $A - B$ are disjoint

$A \cap B \subseteq A \cup (B - C)$

$A - B \subseteq A \cup (B - C)$

(b) $(A \cup B) - C = \{1, 12, 35, 20\}$

(c) $A \cup (B - C) = \{1, 3, 12, 35, 20\}$

$(A \cup B) - C \subseteq A \cup (B - C)$

$A - B \subseteq A \cup (B - C)$

$$\begin{aligned}
 5(a) \quad A - A \cap B &= \{x : (x \in A) \wedge \neg(x \in A \cap B)\} \\
 &= \{x : (x \in A) \wedge \neg(x \in A \wedge x \in B)\} \\
 &= \{x : (x \in A) \wedge (\neg(x \in A) \vee \neg(x \in B))\} \\
 &= \{x : x \in A \wedge \neg(x \in B)\} = A - B
 \end{aligned}$$

because $P \wedge (\neg P \vee \neg Q)$ is equiv. to $P \wedge \neg Q$
 (check this by using a truth table)

$$\begin{aligned}
 (b) \quad A \cup (B \cap C) &= \{x : (x \in A) \vee (x \in B \cap C)\} \\
 &= \{x : (x \in A) \vee (x \in B \wedge x \in C)\} \\
 &\xrightarrow{\quad} = \{x : (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\} \\
 &= \{x : x \in A \cup B \wedge x \in A \cup C\} = (A \cup B) \cap (A \cup C)
 \end{aligned}$$

because $P \vee (Q \wedge R)$ is equiv. to $(P \vee Q) \wedge (P \vee R)$.

$$\begin{aligned}
 (c) \quad (A \cup B) - C &= \{x : x \in A \cup B \wedge \neg(x \in C)\} \\
 &= \{x : (x \in A \vee x \in B) \wedge \neg(x \in C)\} \\
 &= \{x : [(x \in A) \wedge \neg(x \in C)] \vee [(x \in B) \wedge \neg(x \in C)]\} \\
 &= \{x : (x \in A - C) \vee (x \in B - C)\} \\
 &= (A - C) \cup (B - C)
 \end{aligned}$$

because $(P \vee Q) \wedge (\neg R)$ is equiv. to $(P \wedge \neg R) \vee (Q \wedge \neg R)$.

$$\begin{aligned}
 3.(a) \quad (A - B) - C &= \{x : [x \in A \wedge \neg(x \in B)] \wedge \neg(x \in C)\} \\
 &= \{x : (x \in A) \wedge \neg(x \in B) \wedge \neg(x \in C)\} \\
 &= \{x : (x \in A) \wedge \neg[(x \in B) \vee (x \in C)]\}
 \end{aligned}$$

A(5)

Ch. 1.4 #8(b) $A - (B - C) = \{x : (x \in A) \wedge \neg(x \in B - C)\}$

$$= \{x : (x \in A) \wedge \neg[(x \in B) \wedge \neg(x \in C)]\}$$

$$= \{x : (x \in A) \wedge [\neg(x \in B) \vee (x \in C)]\}$$

$$= \{x : [(x \in A) \wedge \neg(x \in B)] \vee [(x \in A) \wedge (x \in C)]\}$$

$$\simeq (A - B) \cup (A \cap C).$$

(c) $(A - B) \cup (A \cap C) = \{x : x \in (A - B) \vee x \in (A \cap C)\}$

$$= \{x : [(x \in A) \wedge \neg(x \in B)] \vee [(x \in A) \wedge (x \in C)]\}$$

$$= A - (B - C)$$

(d) $(A - B) \cap (A - C) = \{x : (x \in A - B) \wedge (x \in A - C)\}$

$$= \{x : [(x \in A) \wedge \neg(x \in B)] \wedge [(x \in A) \wedge \neg(x \in C)]\}$$

$$= \{x : (x \in A) \wedge \neg(x \in B) \wedge \neg(x \in C)\}$$

$$\simeq (A - B) - C$$

(e) $A - (B \cup C) = \{x : (x \in A) \wedge \neg(x \in B \cup C)\}$

$$= \{x : (x \in A) \wedge \neg[(x \in B) \vee (x \in C)]\}$$

$$= (A - B) - C$$

So (a) = (d) = (e) and (b) = (c).

Extra: Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then

$$(A \cup B) - B = \{1, 2, 3\} - \{2, 3\} = \{1\} \neq \{1, 2\} = A$$

- Ch. 1.5 #1 (a) $(U \wedge \neg E) \rightarrow \neg H$ $H = \text{gas is hydrogen}$
- (b) $(F \wedge H) \rightarrow D$ $F = \text{fever}, H = \text{headache}$
- (c) $(F \vee H) \rightarrow D$ $D = \text{George goes to the doctor.}$
- (d) $[\neg(x = z)] \rightarrow [P(x) \rightarrow O(x)]$ $P(x) = x \text{ is prime, } O(x) = x \text{ is odd.}$

A(6)

Ch. 1.5 #2(a) $S_H \rightarrow (G_P \wedge F_A)$ S_H = Mary will sell her house

(d) $[D(x, 4) \vee D(x, 6)] \rightarrow \neg P(x)$. Can also say

 $(F \vee S) \rightarrow \neg P$ $F = x \text{ is divisible by } 4$

(b) $G_M \rightarrow (G_{CH} \wedge A_{DP})$ G_{CH} = good credit history

(c) $\neg S_H \rightarrow K_H$, S_H = someone stops him, K_H = kill himself.

3 (a) $R \rightarrow (W \wedge \neg S)$

(b) $[W \wedge \neg S] \rightarrow R$ converse of (a)

(c) $R \rightarrow [W \wedge \neg S]$ equivalent to (a)

(d) $[W \wedge \neg S] \rightarrow R$ converse of (a)

(e) $[S \vee \neg W] \rightarrow \neg R$ equivalent to (a)

(f) $(R \rightarrow W) \wedge [R \rightarrow \neg S]$ equivalent to (a)

(g) $(W \rightarrow R) \vee [\neg S \rightarrow R]$ converse of (a)

(h) $(W \rightarrow R) \wedge [\neg S \rightarrow R]$ neither

4 (a) $(S \vee E) \wedge (S \rightarrow H) \wedge (E \rightarrow \neg H) \quad \therefore \neg(S \wedge E)$ - valid

Just use truth tables to check that

$[(S \vee E) \wedge (S \rightarrow H) \wedge (E \rightarrow \neg H)] \rightarrow \neg(S \wedge E)$ is a tautology

(You can also assume $S \wedge E$ and then get $H \wedge \neg H$)

This is just like doing a proof by contradiction - but I guess we have to use truth tables because that is what the instructions say.)

6 (b) $(T \wedge U \rightarrow R) \wedge (G \rightarrow \neg R) \wedge (G \wedge T) \quad \therefore \neg U$ - valid7 (c) $[L \leftrightarrow (H \wedge C)] \wedge \neg C \quad \therefore (L \leftrightarrow H)$ - not valid

8 5. (a)(b). Just check that the truth tables are the same.

9 8. $P \wedge Q$ is equiv. to $\neg(P \rightarrow \neg Q)$

Ch. 2.1 #1 (a) $(\forall x)[(\exists y)\{F(x, y)\} \rightarrow S(x)]$

✓ 1 (b) $\neg(\exists x)[C(x) \wedge (\forall y)(D(y) \rightarrow S(x, y))]$

2 (a) $(\forall x)[C(\text{Rolls}, x) \rightarrow (\exists y)(R(y) \wedge U(y, x))]$

2 (b) $(\exists x)[D(x) \wedge M(x)] \rightarrow (\forall y)\{(\exists z)[F(y, z) \wedge D(z)] \rightarrow Q(y)\}$

1 (c) $(\forall x)[(x \neq \text{Mary}) \rightarrow L(x, \text{Mary})] \wedge \neg L(\text{Mary}, \text{Mary})$

1 (d) $(\exists x)[S(\text{Jane}, x) \wedge P(x)] \wedge (\exists y)[S(\text{Roger}, y) \wedge P(y)]$

1 (e) $(\exists x)[P(x) \wedge S(\text{Jane}, x) \wedge S(\text{Roger}, x)]$

2 (d) $[(\exists x) D(x)] \rightarrow D(\text{Jones})$

2 (e) $D(\text{Jones}) \rightarrow (\forall x) D(x)$

3 (a) $(\forall z)[G(z, x) \rightarrow G(z, y)]$

✓ (b) $(\forall a)[(\exists x)(ax^2 + 4x - 2 = 0) \leftrightarrow (a \geq -2)]$

(c) $(\forall x)[(x^3 - 3x < 3) \rightarrow (x < 10)]$

(d) $[(\exists x)(x^2 + 5x = w) \wedge (\exists y)(4 - y^2 = w)] \rightarrow -10 < w < 10$

4 (a) Every unmarried man is unhappy

✓ (b) y is an aunt of x

8 (a) TRUE, Hint: Take $y = 2x$

(b) FALSE, Hint: Take $x = y + 1$

(c) FALSE, Hint: Take $x = 1$

(d) TRUE, Hint: $(y < x) \wedge (x < 10) \rightarrow y < 9$

(e) TRUE, Hint: Take $y = 25$ and $z = 75$

(f) FALSE Hint: Take $x = 100$

9 (a) TRUE Hint: Take $y = 2x$

(b) FALSE Hint: Take $x = (y^2 + 1)/2$. Then

$$2x - y = y^2 + 1 - y = (y - 1/2)^2 + 3/4 \geq 3/4$$

9

A(8)

Ch. 2.1 #8 (c) TRUE . Hint: Take $y = x/2$

(d) FALSE Hint: $(y < x) \wedge (x < 10) \rightarrow y < 9$. Take $x = 9\frac{2}{3}, y = 9\frac{1}{3}$.

(e) TRUE Hint: Take $z = 99$ and $y = 1$

(f) TRUE Hint: Take $y = 100+x$ and $z = -x$

Ch. 2.2 #1(a) There is a math major with a friend who does
✓ not need help with his homework

2(a) Everyone in the freshman class has a room mate

1(b) There is someone with a roommate who likes somebody.

2(b) Someone likes nobody or someone likes everybody

2(c) $(\exists a \in A)(\forall b \in B)[\neg(a \in C \leftrightarrow b \in C)]$

1(c) $\neg(\forall x)[x \in A \cup B \rightarrow x \in C - D]$ which is equiv. to
 $\neg(\forall x)[x \notin A \cup B \vee x \in C - D]$ which is equiv. to
 $(\exists x)[x \in A \cup B \wedge x \notin C - D]$

2(d) $\neg(\forall y)[(y > 0) \rightarrow (\exists x)(ax^2 + bx + c = y)]$ equiv. to

✓ $\neg(\forall y)[(y > 0) \vee (\exists x)(ax^2 + bx + c = y)]$ equiv. to
 $(\exists y)[y > 0 \wedge \neg(\exists x)(ax^2 + bx + c = y)]$ equiv. to
 $(\exists y)[y > 0 \wedge (\forall x)(ax^2 + bx + c \neq y)]$

3 (a) TRUE

✓ (b) FALSE . Take $x = 1$ or 7

(c) TRUE Could take $x = -1$ or 9 , but $-1 \notin \mathbb{N}$

(d) TRUE Take $x = 9$ and $y = 9$

4 $\neg(\exists x)Q(x) \approx (\forall x)\neg Q(x)$

FIRST Quant. | Neg
F.Q.N.L | Q.W.

✓ $\neg\neg(\exists x)Q(x) \approx \neg(\forall x)\neg Q(x)$

negating both sides

$\therefore (\exists x)Q(x) \approx \neg(\forall x)\neg Q(x)$

by double negation

A(9)

Ch. 2.2 #4 ✓ ∴ $(\exists x) \neg P(x) \approx \neg(\forall x) \neg \neg P(x)$ Replace $\neg Q(x)$ by $\neg \neg P(x)$

∴ $(\exists x) \neg P(x) \approx \neg(\forall x) P(x)$ by double negation

∴ $\neg(\forall x) P(x) \approx (\exists x) \neg P(x)$

5. $\neg(\exists x \in A) P(x) \approx \neg(\exists x) [x \in A \wedge P(x)]$

✓ $\approx (\forall x) \neg[x \in A \wedge P(x)]$

$\approx (\forall x) [\neg(x \in A) \vee \neg P(x)]$

$\approx (\forall x) [x \in A \rightarrow \neg P(x)]$

$\approx (\forall x \in A) \neg P(x)$

6. We know that \forall distributes over conjunction

✓ ∴ $(\forall x) [R(x) \wedge S(x)] \approx (\forall x) R(x) \wedge (\forall x) S(x)$

∴ $\neg(\forall x) [R(x) \wedge S(x)] \approx \neg[(\forall x) R(x) \wedge (\forall x) S(x)]$

∴ $\neg(\forall x) [R(x) \wedge S(x)] \approx \neg(\forall x) R(x) \vee \neg(\forall x) S(x)$

∴ $\neg(\forall x) [\neg P(x) \wedge \neg Q(x)] \approx \neg(\forall x) (\neg P(x)) \vee \neg(\forall x) (\neg Q(x))$

∴ $(\exists x) \neg[\neg P(x) \wedge \neg Q(x)] \approx (\exists x) \neg \neg P(x) \vee (\exists x) \neg \neg Q(x)$

∴ $(\exists x) [P(x) \vee Q(x)] \approx (\exists x) P(x) \vee (\exists x) Q(x)$

7. $(\exists x) [P(x) \rightarrow Q(x)] \approx (\exists x) [\neg P(x) \vee Q(x)]$

✓ $\approx (\exists x) \neg P(x) \vee (\exists x) Q(x)$ by #5

$\approx \neg(\forall x) P(x) \vee \exists x Q(x)$

$\approx (\forall x) P(x) \rightarrow (\exists x) Q(x)$

8. ✓ $(\forall x \in A) P(x) \wedge (\forall x \in B) P(x) \approx (\forall x) [x \in A \rightarrow P(x)] \wedge (\forall x) [x \in B \rightarrow P(x)]$

$\approx (\forall x) \{ [x \in A \rightarrow P(x)] \wedge [x \in B \rightarrow P(x)] \}$

$\approx (\forall x) \{ [x \notin A \vee P(x)] \wedge [x \notin B \vee P(x)] \}$

$\approx (\forall x) \{ [x \notin A \wedge x \notin B] \vee P(x) \}$ distr. law

$\approx (\forall x) \{ \neg(x \in A \cup B) \vee P(x) \}$

$\approx (\forall x) \{ x \notin A \cup B \rightarrow P(x) \} \approx (\forall x \in A \cup B) P(x)$

A (10)

Ch. 2.2 #9(a) No. (b) Let $P(x) = x \text{ is even}$, and
 $\checkmark Q(x) = x \text{ is odd}.$

Then $(\forall x)[P(x) \vee Q(x)]$ is clearly true in \mathbb{N}
but $(\forall x)P(x)$ and $(\forall x)Q(x)$ are both false in \mathbb{N} .

$$\begin{aligned}\text{Ch. 2.3 #1(g)} \quad \mathcal{F} \subseteq \mathcal{P}(A) &\approx (\forall x \in \mathcal{F})(x \in \mathcal{P}(A)) \\ &\approx (\forall x \in \mathcal{F})(x \subseteq A) \\ &\approx (\forall x \in \mathcal{F})(\forall y \in x)(y \in A)\end{aligned}$$

$$\begin{aligned}(b) \quad x \in \cup \mathcal{F} - \cup \mathcal{G} &\approx x \in \cup \mathcal{F} \wedge x \notin \cup \mathcal{G} \\ &\approx (\exists y \in \mathcal{F})(x \in y) \wedge (\forall z \in \mathcal{G})(x \notin z)\end{aligned}$$

$$(c) \quad A \subseteq \{2n+1 : n \in \mathbb{N}\} \approx (\forall x \in A)(\exists n \in \mathbb{N})(x = 2n+1)$$

$$\begin{aligned}(d) \quad \{n^2+n+1 : n \in \mathbb{N}\} &\subseteq \{2n+1 : n \in \mathbb{N}\} \\ (\forall n \in \mathbb{N}) \quad (\exists k \in \mathbb{N}) \quad (n^2+n+1 &= 2k+1)\end{aligned}$$

$$\begin{aligned}(e) \quad \{x \in B : x \notin C\} \in \mathcal{P}(A) &\approx \{x : x \in B \wedge x \notin C\} \subseteq A \\ (\forall x) \quad [(x \in B \wedge x \notin C) \rightarrow x \in A] &\quad\end{aligned}$$

$$(f) \quad x \in \bigcap_{i \in I} (A_i \cup B_i) \approx (\forall i \in I) [x \in (A_i \cup B_i)]$$

$$(g) \quad x \in (\bigcap_{i \in I} A_i) \cup (\bigcap_{i \in I} B_i) \approx (\forall i \in I)(x \in A_i) \vee (\forall i \in I)(x \in B_i)$$

$$\begin{aligned}(h) \quad P(\bigcup_{i \in I} A_i) \neq \bigcup_{i \in I} P(A_i) &\approx (\exists x \in \bigcup_{i \in I} A_i), (x \notin \bigcup_{i \in I} P(A_i)) \\ &\approx (\exists x) [(\forall y \in x)(\exists i \in I)(y \in A_i) \wedge (\forall i \in I)(x \notin A_i)] \\ &\approx (\exists x) [(\forall y \in x)(\exists i \in I)(y \in A_i) \wedge (\forall i \in I)(\exists z \in x)(z \notin A_i)]\end{aligned}$$

$$2 \quad P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

A(11)

Ch. 2.3 #3 (a) $\cap \mathcal{F} = \{\text{red, blue}\}$

(b) $\cup \mathcal{F} = \{\text{red, blue, green, orange, purple}\}$

#4 (a) $A_i = \{i-1, i, i+1, 2i\}$

$I = \{2, 3, 4, 5\}$

$A_2 = \{1, 2, 3, 4\}$

$A_3 = \{2, 3, 4, 6\}$

$A_4 = \{3, 4, 5, 8\}$

$A_5 = \{4, 5, 6, 10\}$

(b) $\bigcap_{i \in I} A_i = \{4\}$

$\bigcup_{i \in I} A_i = \{1, 2, 3, 4, 5, 6, 8, 10\}$

8. (a) $A_2 = \{2, 4\}, B_2 = \{2, 3\}, A_3 = \{3, 6\}, B_3 = \{3, 4\}$

(b) $\bigcap_{i \in I} (A_i \cup B_i) = \{2, 3, 4\} \cap \{3, 4, 6\} = \{3, 4\}$

$(\bigcap_{i \in I} A_i) \cup (\bigcap_{i \in I} B_i) = (\{2, 4\} \cap \{3, 6\}) \cup (\{2, 3\} \cap \{3, 4\}) = \{3\}$

They are not the same

(c) $(\bigcap_{i \in I} A_i) \cup (\bigcap_{i \in I} B_i) \subseteq \bigcap_{i \in I} (A_i \cup B_i)$

10.

$x \in P(A \cap B) \Leftrightarrow (\forall y \in x) (y \in A \cap B)$

$\Leftrightarrow (\forall y \in x) (y \in A \wedge y \in B)$

$\Leftrightarrow (\forall y \in x) (y \in A) \wedge (\forall y \in x) (y \in B)$

$\Leftrightarrow x \in P(A) \wedge x \in P(B)$

The univ. quantifier distributes over conjunction.

11.

Take $A = \{1\}$ and $B = \{2\}$. Then

$P(A) = \{\emptyset, \{1\}\} \text{ and } P(B) = \{\emptyset, \{2\}\}. \text{ So}$

$P(A) \cup P(B) = \{\emptyset, \{1\}, \{2\}\}. \text{ But } P(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

12.

(a) easy. " \exists " distributes over " \vee ".

(b) easy. " \forall " distributes over " \wedge ".

(c) $x \in \bigcap_{i \in I} (A_i - B_i) \Leftrightarrow (\forall i \in I) (x_i \in A_i \wedge x_i \notin B_i)$

$\Leftrightarrow (\forall i \in I) (x_i \in A_i) \wedge (\forall i \in I) (x_i \notin B_i)$

$\Leftrightarrow (\forall i \in I) (x_i \in A_i) \wedge \neg (\exists i \in I) (x_i \in B_i)$

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$$\text{Ch. 2.3 #13 (a)} \quad B_j = A_{1,j} \cup A_{2,j} = \{1, 2, j, 1+j, 2+j\}$$

$$\therefore B_3 = \{1, 2, 3, 4, 5\}, \quad B_4 = \{1, 2, 4, 5, 6\}$$

$$(b) \bigcap_{j \in J} B_j = B_3 \cap B_4 = \{1, 2, 4, 5\} = \bigcap_{j \in J} \left(\bigcup_{i \in I} A_{i,j} \right)$$

$$(c) \bigcap_{j \in J} A_{i,j} = A_{i,3} \cap A_{i,4} = \{i, 3, i+3\} \cap \{i, 4, i+4\} = \begin{cases} \{i, 4\} & (\text{if } i=1) \\ \{3\} & (\text{if } i=2) \end{cases}$$

$$\therefore \bigcup_{i \in I} \left(\bigcap_{j \in J} A_{i,j} \right) = \{1, 2, 4\} \neq \bigcap_{j \in J} \left(\bigcup_{i \in I} A_{i,j} \right)$$

$$(d) x \in \bigcap_{j \in J} \bigcup_{i \in I} A_{i,j}$$

$$\Leftrightarrow (\forall j \in J) (x \in \bigcup_{i \in I} A_{i,j})$$

$$\Leftrightarrow (\forall j \in J) (\exists i \in I) (x \in A_{i,j}) \quad \dots \quad (1)$$

$$x \in \bigcup_{i \in I} \left(\bigcap_{j \in J} A_{i,j} \right)$$

$$\Leftrightarrow (\exists i \in I) (x \in \bigcap_{j \in J} A_{i,j})$$

$$\Leftrightarrow (\exists i \in I) (\forall j \in J) (x \in A_{i,j}) \quad \dots \quad (2)$$

(1) and (2) are not equivalent (in general).

Sometimes it is possible for $\bigcap_{j \in J} \bigcup_{i \in I} A_{i,j}$ to be equal to $\bigcup_{i \in I} \bigcap_{j \in J} A_{i,j}$. For instance if $|I| = |J| = 1$, then they will be equal.

However we always have that (2) \Rightarrow 1.

So $\bigcup_{i \in I} \left(\bigcap_{j \in J} A_{i,j} \right) \subseteq \bigcap_{j \in J} \left(\bigcup_{i \in I} A_{i,j} \right)$, always.