Ch-4.1 # 4 (a) $A \times (B \cap C) = \{1,2,3\} \times (\{1,4\} \cap \{3,4\}) = \{1,2,3\} \times \{4\}$

$= \{\langle 1,4\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$

$(A \times B) \cap (A \times C) = (\{1,2,3\} \times \{1,4\}) \cap (\{1,2,3\} \times \{3,4\})$

$= \{\langle 1,1\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle\}$

$\cap \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,3\rangle, \langle 3,4\rangle\}$

$= \{\langle 1,4\rangle, \langle 2,4\rangle, \langle 3,4\rangle\}$

The rest of the theorem can be verified just as easily

#5 (a) Let $\langle a,b\rangle \in A \times (B \cup C)$. Then $a \in A$ and $b \in B \cup C$. So $a \in A$, and $b \in B$ or $b \in C$. Now if $b \in B$, then $\langle a,b\rangle \in A \times B$. And if $b \in C$, then $\langle a,b\rangle \in A \times C$. $\therefore \langle a,b\rangle \in A \times B$ or $\langle a,b\rangle \in A \times C$. $\therefore \langle a,b\rangle \in (A \times B) \cup (A \times C)$

So $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ ... (1)

Now let $\langle a,b\rangle \in (A \times B) \cup (A \times C)$. Then either $\langle a,b\rangle \in A \times B$ or $\langle a,b\rangle \in A \times C$. In the first case $a \in A$ and $b \in B$ So $a \in A$ and $b \in B \cup C$. So $a \in A \times (B \cup C)$. And in the second case $a \in A$ and $b \in C$. So $a \in A$ and $b \in B \cup C$. So $a \in A \times (B \cup C)$. Thus in either case $\langle a,b\rangle \in A \times (B \cup C)$. Hence $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.. (2)

From (1) & (2) it follows that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b) Let $\langle a,b\rangle \in (A \times B) \cap (C \times D)$. Then $\langle a,b\rangle \in A \times B$ and $\langle a,b\rangle \in C \times D$. $\therefore a \in A$ and $b \in B$, and $a \in C$ and $b \in D$. $\therefore a \in A \cap C$ and $b \in B \cap D$. So $\langle a,b\rangle \in (A \cap C) \times (B \cap D)$. $\therefore (A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.

Now let $\langle a,b\rangle \in (A \cap C) \times (B \cap D)$. Then $a \in A \cap C$ and $b \in B \cap D$. So $a \in A$ and $a \in C$, and $b \in B$ and $b \in D$. So $\langle a,b\rangle \in A \times B$ and $\langle a,b\rangle \in C \times D$. $\therefore (A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$. Hence $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Ch. 4.1 # 6 The first 3 sentences are okay. The third sentence is logically equivalent to

$(x \in A \vee x \in C) \wedge (y \in B \vee y \in D)$

which is logically equivalent to

$(x \in A \wedge y \in B) \vee (x \in A \wedge y \in D) \vee (x \in C \wedge y \in B) \vee (x \in C \wedge y \in D)$.

So we have 4 cases to consider. The "proof" only considers 2 of these 4 cases.

7 $|A \times B| = m \cdot n$

12
10 We want to prove that $(A \times B) \cap (C \times D) = \emptyset$ implies $A \cap C = \emptyset$ or $B \cap D = \emptyset$. It will suffice to prove the contrapositive. Now $\neg(A \cap C = \emptyset \vee B \cap D = \emptyset)$ is equivalent to $A \cap C \neq \emptyset$ and $B \cap D \neq \emptyset$.

So suppose $A \cap C \neq \emptyset$ and $B \cap D \neq \emptyset$. Then we can an element $a \in A \cap C$ and an element $b \in B \cap D$.

So $\langle a,b \rangle \in A \times B$ because $a \in A$ and $b \in B$

and $\langle a,b \rangle \in C \times D$ because $a \in C$ and $b \in D$.

Hence $\langle a,b \rangle \in (A \times B) \cap (C \times D)$ and so $(A \times B) \cap (C \times D) \neq \emptyset$.

So $\neg(A \cap C = \emptyset \vee B \cap D = \emptyset)$ implies $(A \times B) \cap (C \times D) \neq \emptyset$

$\therefore$ $(A \times B) \cap (C \times D) = \emptyset$ implies $A \cap C = \emptyset$ or $B \cap D = \emptyset$.

Ch. 4.2 # 1. (a) domain = set of all parents with a living offspring

range = set of all offsprings with a living parent

2 (a) domain = set of all males with a living sibling

range = set of people with a living brother.

1 (b) domain = $\mathbb{R}$, range = $\mathbb{R}^+ = (0, \infty)$

2 (b) domain = $(-\infty, -1] \cup [1, \infty)$, range = $(-1, 1)$

Ch. 4.2 #5

$R = \{\langle 1,4\rangle, \langle 1,5\rangle, \langle 2,5\rangle, \langle 3,6\rangle\}$

$S = \{\langle 4,5\rangle, \langle 4,6\rangle, \langle 5,4\rangle, \langle 6,6\rangle\}$

(a) $S \circ R = \{\langle 1,5\rangle, \langle 1,6\rangle, \langle 1,4\rangle, \langle 2,4\rangle, \langle 3,6\rangle\}$

(b) $S \circ S = \{\langle 4,4\rangle, \langle 5,5\rangle, \langle 4,6\rangle\}$

(c) $S^{-1} \circ R = \{\langle 1,5\rangle, \langle 1,4\rangle, \langle 2,4\rangle, \langle 3,6\rangle, \langle 3,4\rangle\}$

(d) $R^{-1} \circ S = \{\langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 5,1\rangle, \langle 6,3\rangle\}$

$= (S^{-1} \circ R)^{-1}$

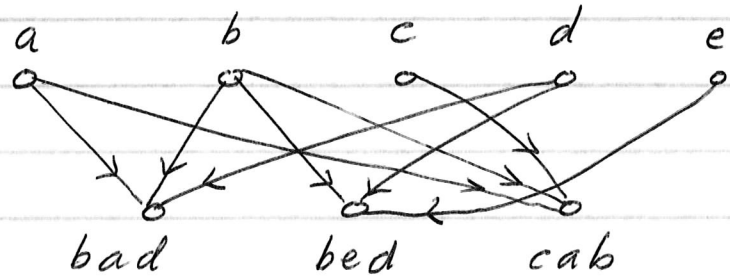
#8 $pEq \wedge qEr \rightarrow pFr$

#10 (a) TRUE. Let $\langle a,b\rangle \in R$. Then $a \in dom(R)$ and $b \in ran(R)$. So $\langle a,b\rangle \in dom(R) \times ran(R)$. Since this is true for any $\langle a,b\rangle \in R$, it follows that $R \subseteq dom(R) \times ran(R)$

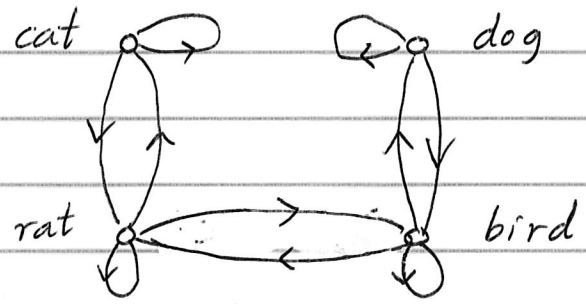
(b) TRUE. Let $\langle b,a\rangle \in R^{-1}$. Then $\langle a,b\rangle \in R$. So $\langle a,b\rangle \in S$ because $R \subseteq S$. Hence $\langle b,a\rangle \in S^{-1}$. Hence $R^{-1} \subseteq S^{-1}$.

(c) Let $\langle b,a\rangle \in (R \cup S)^{-1}$. Then $\langle a,b\rangle \in R \cup S$. So $\langle a,b\rangle \in R$ or $\langle a,b\rangle \in S$. In the first case $\langle b,a\rangle \in R^{-1}$ and in the second case $\langle b,a\rangle \in S^{-1}$. So $\langle a,b\rangle \in R^{-1} \cup S^{-1}$. Hence $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Ch. 4.3 #1



Ch. 4.3 # 2 (a)



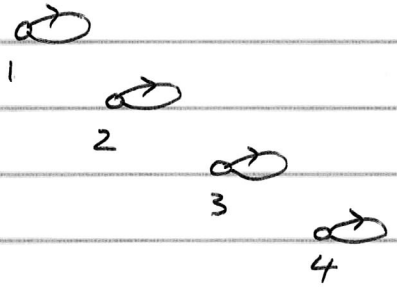
(b) R is reflexive

(c) R is symmetric

(d) R is not transitive

$\langle rat, bird\rangle \in R \wedge \langle bird, dog\rangle \in R$ but $\langle rdt, dog\rangle \notin R$.

3. $i_A =$



4. (a) $R_1 = \{\langle a,c\rangle, \langle b,d\rangle, \langle c,c\rangle, \langle d,a\rangle, \langle d,b\rangle\}$

not reflexive, not symmetric, not transitive

$\langle a,a\rangle \notin R_1$; $\langle a,c\rangle \in R_1$ but $\langle c,a\rangle \notin R_1$; $\langle d,a\rangle \in R_1 \wedge \langle a,c\rangle \in R_1$ but $(d,c) \notin R_1$

(b) $R_2 = \{\langle a,b\rangle, \langle a,d\rangle, \langle b,a\rangle, \langle b,d\rangle\}$

not reflexive, not symmetric, not transitive

$\langle a,a\rangle \in R_2$; $\langle a,d\rangle \in R_2$ but $\langle d,a\rangle \notin R_2$; $\langle a,b\rangle \in R_2 \wedge \langle b,a\rangle \in R_2$ but $\langle a,a\rangle \notin R_2$

(c) $R_3 = \{\langle a,a\rangle, \langle b,b\rangle, \langle b,d\rangle, \langle c,c\rangle, \langle d,b\rangle, \langle d,d\rangle\}$

reflexive, symmetric, transitive

(d) $R_4 = \{\langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle b,c\rangle, \langle b,d\rangle\}$

not reflexive, not symmetric, transitive.

$\langle a,a\rangle \notin R_4$; $\langle a,c\rangle \in R_4$ but $\langle c,a\rangle \notin R_4$.

Ch-4.3 #5. $S \circ R = \{\langle a,y\rangle, \langle a,z\rangle, \langle b,x\rangle, \langle c,y\rangle, \langle c,z\rangle\}$

14(a) Suppose R and S are reflexive on A. Then $i_A \in R$ and $i_A \in S$. So $i_A \subseteq R \cap S$
Thus $R \cap S$ will be reflexive.

13(a) Also $i_A \subseteq R \cup S$, so $R \cup S$ will be reflexive

16(a) Now $i_A = i_A \circ i_A \subseteq R \circ S$, So $R \circ S$ will

12(a) also be reflexive. Finally $i_A \subseteq R^{-1}$. $\therefore R^{-1}$ is reflexive

14(b) Suppose R and S are symmetric. Then
$\langle a,b\rangle \in R \rightarrow \langle b,a\rangle \in R$ and $\langle a,b\rangle \in S \rightarrow \langle b,a\rangle \in S$.
Let $\langle a,b\rangle \in R \cap S$ Then $\langle a,b\rangle \in R$ and $\langle a,b\rangle \in S$. So $\langle b,a\rangle \in R$ and $\langle b,a\rangle \in S$. So $\langle b,a\rangle \in R \cap S$. Hence $R \cap S$ is symmetric.

13(b) Similarly if $\langle a,b\rangle \in R \cup S$, then $\langle a,b\rangle \in R$ or $\langle a,b\rangle \in S$. In either case we get $\langle b,a\rangle \in R \cup S$. So $R \cup S$ is also symmetric.

Now suppose $\langle a,b\rangle \in R \circ S$. Then $\exists c \in A$ such that $\langle a,c\rangle \in S$ and $\langle c,b\rangle \in R$. So $\langle c,a\rangle \in S$ and $\langle b,c\rangle \in R$. $\therefore \langle b,a\rangle \in S \circ R$. So it seems that we can't prove that $\langle b,a\rangle \in R \circ S$. $\therefore$ Let us look for a counter example.

16(b) Let $S = \{\langle 1,1\rangle\}$ and $R = \{\langle 1,2\rangle, \langle 2,1\rangle\}$. Then $R \circ S = \{\langle 1,2\rangle\}$. R and S are symmetric but $R \circ S$ is not symmetric.

12(b) Finally R is symmetric iff $R = R^{-1}$. Since R is symm., $R = R^{-1}$. $\therefore R^{-1} = (R^{-1})^{-1}$. $\therefore R^{-1}$ is symm.

Ch. 4.3 NOTE: Suppose R and S are transitive. Then
$\langle a,b\rangle \in R \wedge \langle b,c\rangle \in R \rightarrow \langle a,c\rangle \in R$ and
$\langle a,b\rangle \in S \wedge \langle b,c\rangle \in S \rightarrow \langle a,c\rangle \in S$.

14(c) Suppose $\langle a,b\rangle \in R\cap S \wedge \langle b,c\rangle \in R\cap S$. Then $\langle a,b\rangle \in R$ and $\langle b,c\rangle \in R$. So $\langle a,c\rangle \in R$. Also $\langle a,b\rangle \in S$ and $\langle b,c\rangle \in S$. So $\langle a,c\rangle \in S$. $\therefore \langle a,c\rangle \in R\cap S$ So $R\cap S$ will be transitive.

13(c) Let $R = \{\langle 1,2\rangle\}$ and $S = \{\langle 2,3\rangle\}$. Then R and S are transitive. But $R\cup S = \{\langle 1,2\rangle, \langle 2,3\rangle\}$ is not transitive because $\langle 1,2\rangle \in R\cup S$ and $\langle 2,3\rangle \in R\cup S$ but $\langle 1,3\rangle \notin R\cup S$.

12(c) Suppose $\langle c,b\rangle \in R^{-1}$ and $\langle b,a\rangle \in R^{-1}$. Then $\langle a,b\rangle \in R$ and $\langle b,c\rangle \in R$. So $\langle a,c\rangle \in R$ because R is transitive. Hence $\langle c,a\rangle \in R^{-1}$. So $\langle c,b\rangle \in R^{-1}$ and $\langle b,a\rangle \in R^{-1}$ implies $\langle c,a\rangle \in R^{-1}$. $\therefore R^{-1}$ is transitive

16(c) Finally let $S = \{\langle 1,2\rangle, \langle 3,4\rangle\}$ and $R = \{\langle 2,3\rangle, \langle 4,5\rangle\}$ We can then check that S and R are transitive. Also $R\circ S = \{\langle 1,3\rangle, \langle 3,5\rangle\}$. So $R\circ S$ is not transitive because $\langle 1,3\rangle \in R\circ S$ and $\langle 3,5\rangle \in R\circ S$ but $\langle 1,5\rangle \notin R\circ S$. Hence if R and S are transitive, it does not follow that $R\circ S$ is transitive.

Qu#16. Suppose R and S are relations. Prove or give a counter-example
(a) If R and S are reflexive, must $R\circ S$ also be reflexive
(b) If R and S are symmetric, must $R\circ S$ also be symmetric
(c) If R and S are transitive, must $R\circ S$ also be transitive.

Ch 4.6 # 1. $P_1 = \{\{1\}, \{2\}, \{3\}\}$, $P_2 = \{\{1\}, \{2,3\}\}$, $P_3 = \{\{2\}, \{1,3\}\}$, $P_4 = \{\{3\}, \{1,2\}\}$, $P_5 = \{\{1,2,3\}\}$. Note each P_i is related to the corresponding equiv. relation R_i below, $(i = 1, \ldots, 5)$.

2 $R_1 = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle\}$, $R_2 = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

$R_3 = \{\langle 2,2\rangle, \langle 1,1\rangle, \langle 1,3\rangle, \langle 3,1\rangle, \langle 3,3\rangle\}$

$R_4 = \{\langle 3,3\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\}$

$R_5 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 3,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle\}$

3 (a) R is an equivalence relation

C_a = set of all words beginning with "a"

C_b = set " " with "b"

⋮ ⋮ ⋮ ⋮

C_z = set " " with "z"

The equivalence classes are $C_a, C_b, \ldots, C_z$.

3 (c) T is an equiv. relation; C_n = words with n letters.

3 (b) S is not an equivalence relation because S is not transitive. $\langle go, to\rangle \in S$ and $\langle to, at\rangle \in S$ but $\langle go, at\rangle \notin S$.

4 (a) R is not an equiv. relation because it is not symmetric.

4 (b) $S = \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Q}\}$ is an equivalence relation. Verify that R is reflexive, symmetric and transitive. The equivalence classes are $C_x = \{x + q : q \in \mathbb{Q}\}$. For each $x \in \mathbb{R}$, C_x will be an equivalence class — but some of these will coincide. For example

$$C_{\sqrt{2}+1} = C_{\sqrt{2}-3} = C_{\sqrt{2}}.$$

$$C_0 = C_1 = C_{1/2} = C_{3/4} = C_{2/3}.$$

Ch. 4.6 #4 (c) $T = \{(x,y) \in \mathbb{R}\times\mathbb{R} : (\exists n\in\mathbb{Z})(y = x\cdot 10^n)\}$

For each $x \in \mathbb{R}$, $x = x\cdot 10^0$. So $(x,x)\in R$ for each $x\in\mathbb{R}$. $\therefore$ T is reflexive

Suppose $(x,y)\in T$. Then $(\exists n\in\mathbb{Z})(y = x\cdot 10^n)$. So $x = y\cdot 10^{-n}$. Hence $(y,x)\in T$. So T is symmetric.

Suppose $(x,y)\in T$ and $(y,z)\in T$. Then $(\exists n\in\mathbb{Z})(y = x\cdot 10^n)$ and $(\exists k\in\mathbb{Z})(z = y\cdot 10^k)$. So $z = y\cdot 10^k = (x\cdot 10^n)\cdot 10^k = x\cdot 10^{n+k}$. Hence $(x,z)\in T$. So T is transitive.

$\therefore$ T is an equivalence relation

Let $C_x = \{x\cdot 10^n : n\in\mathbb{Z}\}$. Then for each $x\in\mathbb{R}$, C_x will be an equivalence class. Some of these classes will coincide. For example

$$C_1 = C_{10} = C_{100} = C_{1/10} = C_{1/100} = C_{1/1000}$$

$$C_{\sqrt{2}} = C_{10\sqrt{2}} = C_{100\sqrt{2}} = C_{\sqrt{2}/10} = C_{\sqrt{2}/100}.$$

Suppose R and S are equivalence relations. Then R and S are both reflexive, symmetric and transitive.

19 (a) From Problems 4.3 #12-16, it follows that $R\cap S$ will be reflexive, symmetric & transitive. So $R\cap S$ will be an equivalence relation

27 (a) $R\cup S$ will not always be an equiv. relation because $R\cup S$ will not always be transitive.

28 (a) $R\circ S$ will not always be an equiv. relation because $R\circ S$ will not always be symm. or trans.

Ch. 4.6 # 10. (a) Let $x \in \mathbb{Z}$. Then $x - x = 0 = 0 \cdot m$. So $\langle x, x \rangle \in C_m$. So C_m is reflexive.
Suppose $\langle x, y \rangle \in C_m$. Then $(\exists k \in \mathbb{Z})$ such that $x - y = km$. So
$y - x = (-k) \cdot m$
Hence $\langle y, x \rangle \in C_m$. So C_m is symmetric.
It was already shown on p. 221 that $\langle x, y \rangle \in C_m$ and $\langle y, z \rangle \in C_m$ implies $\langle x, z \rangle \in C_m$.
So C_m is also transitive. Hence C_m is an equivalence relation.

(b) C_2 : $[0]_2 = \{2k : k \in \mathbb{Z}\}$ and
$[1]_2 = \{2k+1 : k \in \mathbb{Z}\}$ are the equiv. classes
C_3 : $[0]_3 = \{3k : k \in \mathbb{Z}\}$,
$[1]_3 = \{3k+1 : k \in \mathbb{Z}\}$,
and $[2]_3 = \{3k+2 : k \in \mathbb{Z}\}$ are the equiv. classes
C_m will have m equiv. classes - $[0]_m, \cdots, [m-1]_m$

24(a) Suppose that R is a transitive, reflexive relation on A
Then $i_A \in R$, $i_A \subseteq R^{-1}$. Also $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$.
So $i_A \subseteq R \cap R^{-1}$. Hence $R \cap R^{-1}$ is reflexive.
Also if $(z, y) \in R^{-1}$ and $(y, x) \in R^{-1}$. Then $(x, y) \in R$ and $\langle y, z \rangle \in R$. So $(x, z) \in R$ because R is transitive
Thus $(z, x) \in R^{-1}$. Hence R^{-1} is transitive.
From Problem 4.3 # 9(c), it follows that $R \cap R^{-1}$ is also transitive. Finally suppose $(x, y) \in R \cap R^{-1}$. Then $(x, y) \in R$ and $(x, y) \in R^{-1}$. So $\langle y, x \rangle \in R^{-1}$ and $(x, x) \in (R^{-1})^{-1} = R$
Hence $\langle y, x \rangle \in R \cap R^{-1}$. So $R \cap R^{-1}$ is symm. $\therefore$ $R \cap R^{-1}$ is an E.R.

Ch. 5.1 #1 (a) YES. (b) NO. $\langle 1,2\rangle \in f$ & $\langle 1,3\rangle \in f$ but $2 \neq 3$. (c) YES.

2 (a) NO, $\neg(\exists w \in B)(\langle d, w\rangle \in f)$

(b) f is not a function $\langle g0, g\rangle$ & $\langle g0, 0\rangle \in f$ but $g \neq 0$.
g is a function from W to A.

(c) YES, R is a function from P to P.

#3 (a) $f(a) = b$, $f(b) = b$, $f(c) = a$

(b) $H(\text{Italy}) = \text{Rome}$

(c) $f(2) = 2^2 - 2(2) = 0$

(d) $F(\{1,3\}) = \{1,2,3\} - \{1,3\} = \{2\}$

$L \circ H$ = identity func.

#5 (a) $L \circ H : N \to N$ $\quad (L \circ H)(\text{a country}) = \text{the same country}$

(b) $H \circ L : C \to C$ $\quad (H \circ L)(\text{a city}) = \text{capital of the country in which that city is located}$

#6 $(f \circ g)(x) = f(g(x))$

$= f(2x-1) = \dfrac{1}{(2x-1)^2 + 2} = \dfrac{1}{4x^2 - 4x + 3}$

$(g \circ f)(x) = g(f(x))$

$= g\left(\dfrac{1}{x^2+2}\right) = 2\left(\dfrac{1}{x^2+2}\right) - 1 = \dfrac{-x^2}{x^2+2}$

#8 (a) i_A is clearly an equivalence relation because it is reflexive, symmetric and transitive. i_A is also a function because $(\forall a \in A)(\exists! b \in A)(\langle a,b\rangle \in i_A)$
The unique b is just a.

(b) Suppose R is an equivalence relation on A which is a function. Then $i_A \subseteq R$ because R is reflexive. Now if $R \neq i_A$, then $(\exists a \in A)(\exists b \in A)(a \neq b \wedge \langle a,b\rangle \in R)$. So $\langle a,a\rangle \in R$ and $\langle a,b\rangle \in R$. So R won't be a function. Hence i_A is the only equiv. relation on A which is a function.

Ch. 5.2 # 5 (a) Suppose $f(x) = f(y)$. Then $\frac{x+1}{x-1} = \frac{y+1}{y-1}$.

So $\frac{x-1+2}{x-1} = \frac{y-1+2}{y-1}$. $\therefore 1 + \frac{2}{x-1} = 1 + \frac{2}{y-1}$

$\therefore \frac{2}{x-1} = \frac{2}{y-1}$ $\therefore x-1 = y-1$ $\therefore x = y$

Hence if $f(x) = f(y)$ then $x = y$. So f is one-to-one

Let $b \in \mathbb{R} - \{1\}$. Take $x = (b+1)/(b-1)$. Then

$$f(x) = \frac{\frac{b+1}{b-1} + 1}{\frac{b+1}{b-1} - 1} = \frac{(b+1)+(b-1)}{b-1} \cdot \frac{b-1}{(b+1)-(b-1)}$$

$$= \frac{2b}{2} = b$$

(b) $(f \circ f)(x) = f(f(x)) = f\left(\frac{x+1}{x-1}\right)$

$$= \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{(x+1)+(x-1)}{x-1} \cdot \frac{x-1}{(x+1)-(x-1)}$$

$$= \frac{2x}{2} = x$$

$\therefore (f \circ f)(x) = x$. $\therefore f \circ f = i_A$

8

6 (a) $f(2) = \{y \in \mathbb{R} : y^2 < 2\} = (-\sqrt{2}, \sqrt{2})$

(b) f is not one-to-one because $f(0) = f(-1)$.

$f(0) = \{y \in \mathbb{R} : y^2 < 0\} = \emptyset$, $f(-1) = \{y \in \mathbb{R} : y^2 < -1\} = \emptyset$.

(c) f is not onto because there is no $x \in \mathbb{R}$ such that $f(x) = [1, 2]$.

9

7 (a) $f(\{\{1,2\}, \{3,4\}\}) = \{1,2\} \cup \{3,4\} = \{1,2,3,4\}$

(b) not one-to-one. $f(\{\{1,2\}\}) = f(\{\{1\}, \{2\}\}) = \{1,2\}$

(c) onto. Let $Y \in A$. Take $X = \{Y\} \in B$. Then $f(X) = Y$.

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Ch 5.2 # 9 (a) $f: A \to B$ and $g: B \to C$. Suppose f is onto and g is not one-to-one. The we can find $b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$ and $b_1 \neq b_2$. Since f is onto, we can find $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Also because f is a function and $b_1 \neq b_2$, $a_1 \neq a_2$. So $(g \circ f)(a_1) = g(f(a_1)) = g(b_1) = g(b_2) = (g \circ f)(a_2)$. Since $a_1 \neq a_2$, it follows that $g \circ f$ is not one-to-one.

(b) Suppose f is not onto and g is one-to-one. Then we can find an element $b \in B$ such that there is no $a \in A$ with $f(a) = b$. Let $c = g(b)$. Then there is no element $b' \in B - \{b\}$ such that $g(b') = c$. This is because g is one-to-one. From this it follows that there is no element $a \in A$ such that $(g \circ f)(a) = c$ — because the only way for us to have $g(f(a)) = c$ is for $f(a)$ to be b. But there is no such a. Hence $g \circ f$ is not onto.

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10 (a) If f is one-to-one, then

$$(\forall a_1 \in A)(\forall a_2 \in A)\ [a_1 \neq a_2 \to f(a_1) \neq f(a_2)]$$

Since $C \subseteq A$, this implies that

$$(\forall a_1 \in C)(\forall a_2 \in C)\ [a_1 \neq a_2 \to f(a_1) \neq f(a_2)]$$

$\therefore$ $f \restriction C$ will also be one-to-one.

(b) If $f \restriction C$ is onto, then $(\forall b \in B)(\exists a \in C)\ [f(a) = b]$

Since $C \subseteq A$, it follows that $(\forall b \in B)(\exists a \in A)\ [f(a) = b]$.

$\therefore$ f will also be onto.

(c) Let $A = \mathbb{R}$, $B = \mathbb{R}$, $C = [0, \infty)$. Put $f_1(x) = x^2$. f is not one-to-one

Let $A = \mathbb{R}$, $B = \mathbb{R}$, $C = [0, \infty)$. Put $f_2(x) = x$. $f \restriction C$ is not onto.

Ch. 5.3 #3 (a) Suppose $f(x_1) = f(x_2)$. Then

$$(2x_1+5)/3 = (2x_2+5)/3$$

So $2x_1+5 = 2x_2+5$ $\therefore$ $2x_1 = 2x_2$

$\therefore$ $x_1 = x_2$. Hence f is one-to-one

(b) Let $z \in \mathbb{R}$. Put $x = (3z-5)/2$. Then

$$f(x) = (2x+5)/3 = \left(2 \cdot \frac{3z-5}{2} + 5\right)/3$$

$$= (3z-5+5)/3 = 3z/3 = z$$

$\therefore$ f is onto.

(c) Let $y = f^{-1}(x)$. Then $f(y) = x$. So

$(2y+5)/3 = x$ $\therefore$ $2y+5 = 3x$

$\therefore$ $2y = 3x-5$. So $y = (3x-5)/2$

Hence $f^{-1}(x) = (3x-5)/2$.

#6 (a) Suppose $f(x_1) = f(x_2)$. Then

$$\frac{3x_1}{x_1-2} = \frac{3x_2}{x_2-2}$$

$$\therefore \quad \frac{3(x_1-2)+6}{x_1-2} = \frac{3(x_2-2)+6}{x_2-2} \quad \therefore \quad 3 + \frac{6}{x_1-2} = 3 + \frac{6}{x_2-2}$$

$$\therefore \quad \frac{6}{x_1-2} = \frac{6}{x_2-2} \quad \therefore \quad x_1-2 = x_2-2 \quad \therefore \quad x_1 = x_2$$

Hence f is one-to-one

(b) Take $B = \mathbb{R} - \{3\}$ and let $z \in B$. Put $x = \frac{6}{z-3} + 2$

Then $f(x) = \dfrac{3(6/(z-3)+2)}{[6/(z-3)+2]-2}$

$$= 3 \cdot \frac{[6 + 2(z-3)]}{z-3} \cdot \frac{z-3}{6}$$

$$= \frac{3 \cdot 2z}{6} = \frac{6z}{6} = z$$

$\therefore$ $f : A \to B$ is onto.

(c) $f^{-1}(x) = [6/(x-3)] + 2 = 2x/(x-3)$.

Ch. 5.3 #7 $f(x) = (x+7)/5$, $f_1(x) = x+7$, $f_2(x) = x/5$

(a) $(f_2 \circ f_1)(x) = f_2(f_1(x)) = f_2(x+7) = \frac{x+7}{5} = f(x)$

$\therefore f_2 \circ f_1 = f$

(b) Let $y = f^{-1}(x)$. Then $f(y) = x$. So $\frac{y+7}{5} = x$

$\therefore y+7 = 5x$. So $y = 5x-7$

$\therefore f^{-1}(x) = 5x-7$.

(c) By inspection, $f_1^{-1}(x) = x-7$ and $f_2^{-1}(x) = 5x$

So $(f_1^{-1} \circ f_2^{-1})(x) = f_1^{-1}(f_2^{-1}(x)) = f_1^{-1}(5x)$

$= 5x-7$

$\therefore (f_2 \circ f_1)^{-1}(x) = (f_1^{-1} \circ f_2^{-1})(x) = 5x-7$

So we have verified that $f^{-1} = (f_2 \circ f_1)^{-1}$.

11 (a) Suppose f is one-to-one and $f \circ g = i_B$. Let b be an arbitrary element of B. Put $a = g(b)$. Then $f(a) = f(g(b)) = (f \circ g)(b) = i_B(b) = b$. Hence f is onto. So $f^{-1}: B \to A$ exists. Since $f \circ g = i_B$, it follows that

$f^{-1} \circ (f \circ g) = f^{-1} \circ i_B$

$\therefore (f^{-1} \circ f) \circ g = f^{-1}$

$\therefore i_A \circ g = f^{-1}$. So $g = f^{-1}$.

(b) Suppose f is onto and $g \circ f = i_A$. Suppose $f(x_1) = f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$.

So $(g \circ f)(x_1) = (g \circ f)(x_2)$

$\therefore i_A(x_1) = i_A(x_2)$. So $x_1 = x_2$

Hence f is one-to-one. So $f^{-1}: B \to A$ exists. Since $g \circ f = i_A$, it follows that

$(g \circ f) \circ f^{-1} = i_A \circ f^{-1}$

$\therefore g \circ (f \circ f^{-1}) = f^{-1}$. $\therefore g \circ i_B = f^{-1}$. $\therefore g = f^{-1}$.

Ch. 5.3 #11 (c) Suppose $f \circ g = i_B$ and $g \circ f \neq i_A$.
Let $b \in B$. Put $a = g(b)$. Then
$f(a) = f(g(b)) = (f \circ g)(b) = i_B(b) = b$.
So f is onto. Also if f was one-to-one, then by part (a), it will follow that $g = f^{-1}$, and so we would get $g \circ f = f^{-1} \circ f = i_A$ — a contradiction. Hence f cannot be one-to-one.

Now suppose $g(b_1) = g(b_2)$. Then $f(g(b_1)) = f(g(b_2))$. So $(f \circ g)(b_1) = (f \circ g)\, b_2$
$\therefore \quad i_B(b_1) = i_B(b_2)$. Hence $b_1 = b_2$. So g is one-to-one. Also if g was onto, then by part (b), it will follow that $f = g^{-1}$ and from this we would get $g \circ f = g \circ g^{-1} = i_A$ — a contradiction. Hence g cannot be onto.

Ch. 5.4 #1 (a) YES. Let $b \in f[W \cup X]$. Then $\exists a \in W \cup X$ such that $f(a) = b$. Since $a \in W \cup X$, either $a \in W$ or $a \in X$. In the first case $b \in f[W]$ and in the second case $b \in f[X]$. So $b \in f[W] \cup f[X]$ .... (1)
Now suppose $b \in f[W] \cup f[X]$. Then $b \in f[W]$ or $b \in f[X]$. In the first case $\exists a \in W$ such that $b = f(a)$. And in the second case $\exists a \in X$ such that $b = f(a)$. So in either case $\exists a \in W \cup X$ such that $b = f(a)$. Hence $b \in f[W \cup X]$ .... (2)
From (1) & (2) it follows that $f[W \cup X] = f[W] \cup f[X]$.

(b) NO. Let $f : \{1,2\} \to \{3\}$, $W = \{1\}$ and $X = \{2\}$.
Then $f[W - X] = f[\{1\}] = \{3\} \neq \emptyset = \{3\} - \{3\} = f[W] - f[X]$.

5.5

Ch. 5. #1 (c) NO. Let $f: \{1,2\} \to \{4\}$, $W = \{1\}$ and $X = \{2\}$. Then $f[W] = \{4\}$ and $f[X] = \{4\}$. So $f[W] \subseteq f[X]$ but $W \not\subseteq X$. So $f[W] \subseteq f[X] \not\to W \subseteq X$.

5.5 #2 (a) YES. Let $a \in f^{-1}[Y \cap Z]$. Then $f(a) \in Y \cap Z$. So $f(a) \in Y$ and $f(a) \in Z$. $\therefore a \in f^{-1}[Y]$ and $a \in f^{-1}[Z]$ So $a \in f^{-1}[Y] \cap f^{-1}[Z]$. $\therefore f^{-1}[Y \cap Z] \subseteq f^{-1}[Y] \cap f^{-1}[Z]$. Now let $a \in f^{-1}[Y] \cap f^{-1}[Z]$. Then $a \in f^{-1}[Y]$ and $a \in f^{-1}[Z]$. So $f(a) \in Y$ and $f(a) \in Z$. $\therefore f(a) \in Y \cap Z$. So $a \in f^{-1}[Y \cap Z]$. Hence $f^{-1}[Y] \cap f^{-1}[Z] \subseteq f^{-1}[Y \cap Z]$. $\therefore f^{-1}[Y \cap Z] = f^{-1}[Y] \cap f^{-1}[Z]$.

(b) YES. The proof is similar to the one in 2(a)

(c) YES. The proof is similar to the one in 2(a)

(d) NO. Let $f: \{1\} \to \{2,3,4\}$, $f(1) = 2$, $Y = \{2,3\}$ and $Z = \{2,4\}$. Then $f^{-1}[Y] = \{1\} \subseteq \{1\} = f^{-1}[Z]$ but $Y = \{2,3\} \not\subseteq \{2,4\} = Z$. So $f^{-1}[Y] \subseteq f^{-1}[Z] \not\to Y \subseteq Z$.

5.5 #3. NO. Let $f: \{1,2\} \to \{3\}$, $f(1) = f(2) = 3$. Then take $X = \{1\}$. $f^{-1}[f[X]] = f^{-1}[\{3\}] = \{1,2\} \neq X$. Now suppose $a \in X$. Then $f(a) \in f[X]$. So $a \in f^{-1}[f[X]]$. $\therefore X \subseteq f^{-1}[f[X]]$. So $f^{-1}[f[X]] \supseteq X$ but $x \notin f^{-1}[f[X]]$ in general.

5.5 #4. NO. Let $f: \{1\} \to \{2,3\}$, $f(1) = 2$, and $Y = \{2,3\}$. Then $f[f^{-1}[Y]] = f[\{1\}] = \{2\} \neq Y$. So $f[f^{-1}[Y]] \neq Y$, in general.