

8.1

Ch. 7.1 #12 Find the largest natural  $n$  such that  $1+2+3+\dots+n \leq b-1$ . Then take  $i = b - n(n+1)/2$  and  $j = n+2-i$ . Then  $f(i,j) = b$ . So  $f$  will be surjective. Hence  $f$  is bijective.

To illustrate the process for finding  $(i,j)$ , let's say  $b = 66$ . The largest  $n$  such that  $\sum_{k=1}^n k \leq 66-1$  is 10. So  $i + \sum_{k=1}^{i+j-2} k = (66-55) + \sum_{k=1}^{10} k$

$$\therefore i = 66-55 \text{ and } i+j-2 = 10$$

$$\therefore i = 11 \text{ and } j = 10+2-i = 10+2-11 = 1.$$

$$\text{So } 66 = f(11,1).$$

For another example we can take  $b = 60$ . Then the largest  $n$  such that  $1+2+3+\dots+n \leq 60-1$  is 10. So  $i = 60 - 10(10+1)/2 = 60-55 = 5$ . And  $j = 10+2-5 = 7$ .  $\therefore 60 = f(5,7)$ .

8.1 #15 First observe that if  $A \neq \emptyset$ , then  $A$  is countable iff there is an injective function  $f: A \rightarrow \mathbb{Z}^+$ . This follows from Theorem 7.1.5

Now suppose  $A$  is countable. If  $A = \emptyset$ , then  $B$  must also be  $\emptyset$ , so  $B$  is clearly countable. Also if  $A \neq \emptyset$  and  $B = \emptyset$ , then  $B$  is again countable. So assume  $A \neq \emptyset$ ,  $B \neq \emptyset$  and  $B \subseteq A$ . Since  $A$  is countable, we can find an injective function  $f: A \rightarrow \mathbb{Z}^+$ . Let  $g: B \rightarrow \mathbb{Z}$  be defined by  $g(b) = f(b)$ . Since  $f$  is injective,  $g$  will be injective. So  $B$  will be countable.

8.2

Ch. 7.2 #1(a) Suppose  $\mathbb{R} - \mathbb{Q}$  is countable. Then  $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$  would be countable by Theorem 7.2.1(b). But this contradicts the fact that  $\mathbb{R}$  is uncountable. Hence  $\mathbb{R} - \mathbb{Q}$  is uncountable.

(b) Since  $\mathbb{Q}$  is denumerable we can write  $\mathbb{Q} = \{q_n : n \in \mathbb{Z}^+\}$ . Let  $A = \{q_n : n \in \mathbb{Z}^+\}$  and  $B = \{\sqrt{2} + q_n : n \in \mathbb{Z}^+\}$ . Define  $f: \mathbb{R} \rightarrow \mathbb{R} - \mathbb{Q}$  by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{R} - (A \cup B) \\ \sqrt{2} + q_{2n} & \text{if } x \in A \text{ and } x = q_n \\ \sqrt{2} + q_{2n+1} & \text{if } x \in B \text{ and } x = \sqrt{2} + q_n \end{cases}$$

Then  $f$  is bijective. So  $\mathbb{R} \sim \mathbb{R} - \mathbb{Q}$ . From Theorem 7.1.3 it follows that  $\mathbb{R} - \mathbb{Q} \sim \mathbb{R}$ .

#4. Hint: Suppose  $f: A \rightarrow P(A)$  is a function. We will show that  $f$  is not surjective by finding a set  $D \in P(A)$  such that  $D \notin \text{ran}(f)$ . From this it will follow that there can be no bijection from  $A$  to  $P(A)$ . So  $A \neq P(A)$ . Let  $D = \{x \in A : x \notin f(x)\}$  and proceed as in Theorem 8.2.5.

#5(a) We will use  $\mathcal{F}(X, Y)$  to denote the set of all functions from  $X$  to  $Y$ . Suppose  $A \sim B$  and  $C \sim D$ . Then we can find bijections  $i: A \rightarrow B$  and  $j: C \rightarrow D$ . Let  $k: \mathcal{F}(A, C) \rightarrow \mathcal{F}(B, D)$  be defined as follows. If  $f = \{(a, f(a)) : a \in A\} \in \mathcal{F}(A, C)$ , then let  $k(f) = \{(i(a), j(f(a))) : a \in A\}$ . Then  $k(f) \in \mathcal{F}(B, D)$  and  $k$  is a bijection. So  $\mathcal{F}(A, C) \sim \mathcal{F}(B, D)$

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Ch. 7.2 #5(b) Define  $f: \mathcal{P}(A) \rightarrow \mathcal{I}(A, \{\text{yes, no}\})$  by

$$\checkmark f(X) = c_X \quad \text{where} \quad c_X(a) = \begin{cases} \text{yes} & \text{if } a \in A \\ \text{no} & \text{if } a \notin A. \end{cases}$$

Then for each  $X \in \mathcal{P}(A)$ ,  $c_X$  is a function from  $A$  to  $\{\text{yes, no}\}$ . Now verify that  $f$  is bijective. This will show that  $\mathcal{P}(A) \sim \mathcal{I}(A, \{\text{yes, no}\})$

(c) Define  $f: \mathcal{I}(A \times B, C) \rightarrow \mathcal{I}(A, \mathcal{I}(B, C))$  as follows.

If  $g = \{\langle \langle a, b \rangle, g(a, b) \rangle : \langle a, b \rangle \in A \times B\} \in \mathcal{I}(A \times B, C)$

then let  $f(g) = \text{the function } \{\langle a, g_a \rangle : a \in A\}$

where  $g_a = \{\langle b, g(a, b) \rangle : b \in B\}$ . Notice that

$g_a \in \mathcal{I}(B, C)$  and  $f(g) \in \mathcal{I}(A, \mathcal{I}(B, C))$ . Now

verify that  $f$  is bijective. This will show

that  $\mathcal{I}(A \times B, C) \sim \mathcal{I}(A, \mathcal{I}(B, C))$ .

d) We have to show that  $\mathcal{I}(\mathbb{Z}^+, \mathcal{P}(\mathbb{Z}^+)) \sim \mathcal{P}(\mathbb{Z}^+)$ .

But  $\mathcal{P}(\mathbb{Z}^+) \sim \mathcal{I}(\mathbb{Z}^+, \{\text{yes, no}\})$  by 4(b).

So we really have to show that

$\mathcal{I}(\mathbb{Z}^+, \mathcal{I}(\mathbb{Z}^+, \{\text{yes, no}\})) \sim \mathcal{I}(\mathbb{Z}^+, \{\text{yes, no}\})$ .

Now  $\mathcal{I}(\mathbb{Z}^+, \mathcal{I}(\mathbb{Z}^+, \{\text{yes, no}\})) \sim \mathcal{I}(\mathbb{Z}^+ \times \mathbb{Z}^+, \{\text{yes, no}\})$

by 4(c). So we just have to show that

$\mathcal{I}(\mathbb{Z}^+ \times \mathbb{Z}^+, \{\text{yes, no}\}) \sim \mathcal{I}(\mathbb{Z}^+, \{\text{yes, no}\})$ .

But  $\mathbb{Z}^+ \times \mathbb{Z}^+ \sim \mathbb{Z}^+$  by Problem 7.1 #9. So it follows from 4(a) that

$\mathcal{I}(\mathbb{Z}^+ \times \mathbb{Z}^+, \{\text{yes, no}\}) \sim \mathcal{I}(\mathbb{Z}^+, \{\text{yes, no}\})$

which is what we had to show. Thus

$\mathcal{I}(\mathbb{Z}^+, \mathcal{P}(\mathbb{Z}^+)) \sim \mathcal{P}(\mathbb{Z}^+)$ .

8.2

Ch. 7.2 #7 Hint: Let  $f: P(A \cup B) \rightarrow P(A) \times P(B)$  be defined by  $f(X) = \langle A \cap X, B \cap X \rangle$ . Then show that  $f$  is bijective.

8.3

Ch. 7.3 #1. (a) Since  $i_A: A \rightarrow A$  is an injective function, it follows that  $A \leq A$

(b) Suppose  $A \leq B$  and  $B \leq C$ . Then we can find injective functions  $f: A \rightarrow B$  and  $g: B \rightarrow C$ . Now  $g \circ f: A \rightarrow C$  is an injective function by Theorem 5.2.5. So  $A \leq C$ .

8.3 #3 Suppose  $A \subseteq B \subseteq C$  and  $A \sim C$ . Then we can find a bijective function  $f: A \rightarrow C$ . Hence  $f'$  is bijective. So  $f'$  is injective from  $C$  to  $A$ . Since  $A \subseteq B$ ,  $f': C \rightarrow B$  is also an injective function. Thus  $C \leq B$ . Also since  $B \subseteq C$  and  $i_B: B \rightarrow C$  is injective, it follows that  $B \leq C$ . So  $B \leq C$  and  $C \leq B$ . Hence by the Cantor-Schröder-Bernstein Theorem we get  $B \sim C$ .

8.3 #4 We know that  $A \not\leq B$  and  $C \not\leq D$ . So we can find injective functions  $g: A \rightarrow B$  and  $h: C \rightarrow D$ .

(a) Let  $f: A \times C \rightarrow B \times D$  be defined by

$$f(a, c) = \langle g(a), h(c) \rangle.$$

Then  $f$  is an injective function. So  $A \times C \not\leq B \times D$ .

(b) Suppose  $A \cap C = \emptyset$  and  $B \cap D = \emptyset$ . Let

$$f: A \cup C \rightarrow B \cup D, \quad f(x) = \begin{cases} g(x) & \text{if } x \in A \\ h(x) & \text{if } x \in C \end{cases}$$

Then  $f$  is injective. Hence  $A \cup C \not\leq B \cup D$ .

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Ch. 7.3 #4(c) Let  $f: P(A) \rightarrow P(B)$  be defined by

$$f(X) = \{g(a) : a \in X\}. \text{ Then } f(X) \in P(B).$$

Also  $f$  is injective. So  $P(A) \leq P(B)$ .

8.3 #9  $f: A \rightarrow B, f(x) = x/2 \quad g: B \rightarrow A, g(x) = x$

$$A = (0,1], B = (0,1), R = \text{ran}(g) = (0,1)$$

$$A_1 = A - R = (0,1] - (0,1) = \{1\}$$

$$A_2 = g[f[A_1]] = \{g(f(1))\} = \{1/2\}$$

$$A_3 = g[f[A_2]] = \{g(f(1/2))\} = \{1/4\}$$

$$A_{n+1} = g[f[A_n]] = g(f(z^{-n})) = z^{-n}$$

$$X = \bigcup_{n \in \mathbb{Z}^+} A_n = \{z^{-n} : n \in \mathbb{Z}^+\} = \{z^{-n} : n \in \mathbb{N}\}$$

$$h: A \rightarrow B, h(x) = \begin{cases} f(x) & \text{if } x \in X \\ g(x) & \text{if } x \in A - X \end{cases}$$

$$\therefore h(x) = \begin{cases} z^{-n}/2 & \text{if } x = z^{-n} \\ x & \text{if } x \in A - X \end{cases} = \begin{cases} z^{-n-1} & \text{if } x = z^{-n} \\ x & \text{if } x \in A - X \end{cases}$$

8.3 #12.(b) Let  $g: \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  be defined by  $g(x) = (x, 0)$ .

Then  $g$  is injective. So  $\mathbb{R} \leq \mathbb{R} \times \mathbb{R}$ . Now

let  $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x, y) =$  the decimal expansion of  $x$  and  $y$  interlaced. If

$$x = a_n a_{n-1} \dots a_0 \cdot b_1 b_2 b_3 \dots b_k \dots \text{ and}$$

$$y = c_n c_{n-1} \dots c_0 \cdot d_1 d_2 d_3 \dots d_k \dots, \text{ then}$$

$$f(x, y) = a_n c_n a_{n-1} c_{n-1} \dots a_0 c_0 \cdot b_1 d_1 b_2 d_2 \dots b_k d_k \dots$$

Then  $f$  is injective. Here  $n$  is the smallest non-negative integer such that  $\max(x, y) < 10^n$ . So

$\mathbb{R} \times \mathbb{R} \leq \mathbb{R}$ . Hence by the Cantor-Schröder-Bernstein theorem, it follows that  $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$ .

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~~7.3 #12(a)~~ We know from Exercise 7.2 #7 that if  $A$  and  $B$  are disjoint then  $P(A \cup B) \sim P(A) \times P(B)$ . Recall also that if  $A \sim B$ , then  $P(A) \sim P(B)$  by Exercise 7.1 #5.

Now suppose  $A \times A \sim A$  and  $A$  has at least two elements. Since  $A \times \{0\} \cup A \times \{1\} = A \times \{0,1\}$  and  $A$  has at least 2 elements, it follows that

$$\begin{aligned} A &\leq A \times \{0\} \cup A \times \{1\} \leq A \times A \\ \therefore P(A) &\leq P(A \times \{0\} \cup A \times \{1\}) \leq P(A \times A) \end{aligned}$$

Since  $A \times \{0\}$  and  $A \times \{1\}$  are disjoint we get

$$P(A) \leq P(A \times \{0\}) \times P(A \times \{1\}) \leq P(A \times A) \text{ by 7.2#7}$$

But  $A \times \{0\} \sim A \times \{1\} \sim A$  and  $A \times A \sim A$ . So

$$\begin{aligned} P(A) &\leq P(A) \times P(A) \leq P(A) \text{ by 7.1#5} \\ \therefore P(A) &\approx P(A) \times P(A) \text{ if } A \text{ has at} \\ &\text{least two elements and } A \times A \sim A. \end{aligned}$$

(b) We know by Theorem 7.3.3 that  $\mathbb{R} \sim \mathcal{P}(\mathbb{Z}^+)$ .

$$\text{So } \mathbb{R} \times \mathbb{R} \sim \mathcal{P}(\mathbb{Z}^+) \times \mathcal{P}(\mathbb{Z}^+)$$

$$\sim \mathcal{P}(\mathbb{Z}^+) \text{ because } \mathbb{Z}^+ \times \mathbb{Z}^+ \sim \mathbb{Z}^+$$

$$\sim \mathbb{R}, \quad \text{and } |\mathbb{Z}^+|/\mathbb{Z}$$

Thus  $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$ .

END.