MAA 3200 - INTROD TO ADV. MATH FLORIDA INT'L UNIV.

TEST #1 - Fall 2004 TIME: 75 MIN.

Answer all 6 questions. An unjustified answer will receive little or no credit. Begin each question on a separate page.

(15) 1. Translate the following argument into symbolic language and then use a truth table to determine if it is valid. Either Amy or Bob is lying. Either Bob or Cathy is not lying. $\therefore$ if Cathy is lying, then Amy is also lying.

(15) 2(a) Show that $\neg(\exists a\in A)(\exists b\in A)\,[f(a)=f(b)\wedge a\neq b]$ is logically equivalent to $(\forall a\in A)(\forall b\in A)\,[f(a)=f(b)\rightarrow a=b]$.

(b) Let $f:\mathbb{R}-\{2\}\rightarrow\mathbb{R}$, $f(x)=\dfrac{3x}{x-2}$. Is f injective?

(15) 3(a) Let $\mathcal{F}$ be a family of sets. Define what is $\cup\mathcal{F}$

(b) Suppose $A\nsubseteq B$ and $A-C\subseteq B$. Does it follow that $A\cap C\neq\emptyset$? (Give a proof or give a counter-example)

(15) 4(a) Let R be a relation from A to B and S be a relation from B to C. Define what are R^{-1} and $S\circ R$.

(b) Let $f:\mathbb{R}-\{3\}\rightarrow\mathbb{R}-\{2\}$, $f(x)=\dfrac{2x+1}{x-3}$. Find $f^{-1}(x)$.

(20) 5(a) Define what is an equivalence relation on A. Let $a\in A$. Define what is the equivalence class $[a]$.

(b) Let R be the relation on $\mathbb{Z}$ defined by aRb if a^2-b^2 is a multiple of 6. Prove that R is an equivalence relation on $\mathbb{Z}$ and find the equivalence classes of R.

(20) 6(a) Define what is a function from A to B.

(b) If $f:A\rightarrow B$ and $g:B\rightarrow C$ are functions, prove that $g\circ f$ is a function from A to C.

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SOLUTIONS TO TEST # 1 — FALL 2004

1. (a) Let A = Amy is lying, B = Bob is lying, C = Cathy is lying

$(A \vee B) \wedge (\neg B \vee \neg C) \quad \therefore \quad (C \to A)$

(b) $(A \vee B) \wedge (\neg B \vee \neg C) \to (C \to A)$

A	B	C	$(A \vee B)$	$\wedge$	$(\neg B \vee \neg C)$	$\to$	$(C \to A)$
T	T	T	T	F	F	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	T	T	T
F	T	T	T	F	F	T	F
F	T	F	T	T	T	T	T
F	F	T	F	F	T	T	F
F	F	F	F	F	T	T	T

Since $(A \vee B) \wedge (\neg B \vee \neg C) \to (C \to A)$ is a tautology, the argument is valid.

2. (a) $\neg (\exists a \in A)(\exists b \in A)[f(a) = f(b) \wedge a \neq b]$

iff $(\forall a \in A) \neg (\exists b \in A)[f(a) = f(b) \wedge a \neq b]$ by quantifier neg. laws

iff $(\forall a \in A)(\forall b \in A) \neg [f(a) = f(b) \wedge a \neq b]$ " " "

iff $(\forall a \in A)(\forall b \in A)[f(a) \neq f(b) \vee a = b]$ by De Morgans law

iff $(\forall a \in A)(\forall b \in A)[f(a) = f(b) \to a = b]$ bec. $\neg P \vee Q \approx P \to Q$.

(b) Suppose $f(a) = f(b)$. Then $\frac{3a}{a-2} = \frac{3b}{b-2}$. So $3a(b-2) = 3b(a-2)$. Hence $3ab - 6a = 3ab - 6b$ $\therefore$ $6a = 6b$ So $a = b$. Hence f is an injective function.

3 (a) $\bigcup \mathcal{F} = \{x : (\exists A)(A \in \mathcal{F} \wedge x \in A)\}$

(b) Suppose $A \nsubseteq B$ and $A - C \subseteq B$. Since $A \nsubseteq B$, we can find an $x_0 \in A$ with $x_0 \notin B$. Now suppose $x_0 \notin C$. Then $x_0 \in A$ and $x_0 \notin C$. So $x_0 \in A - C$. $\therefore$ $x_0 \in B$ because $A - C \subseteq B$. But this contradicts the fact that $x_0 \notin B$. Hence $x_0 \in C$. So $x_0 \in A$ and $x_0 \in C$. $\therefore$ $x_0 \in A \cap C$. Thus $A \cap C \neq \emptyset$.

4 (a) $R^{-1} = \{\langle b, a\rangle : \langle a, b\rangle \in R\}$

$S \circ R = \{\langle a, c\rangle : (\exists b \in B)(\langle a, b\rangle \in R \text{ and } \langle b, c\rangle \in S)\}$

(b) Let $y = f^{-1}(x)$. Then $f(y) = x$. So

So $\dfrac{2y+1}{y-3} = x$ $\therefore$ $2y + 1 = xy - 3x$

$\therefore$ $3x + 1 = xy - 2y$ $\therefore$ $3x+1 = y(x-2)$

$\therefore$ $y = \dfrac{3x+1}{x-2}$. $\therefore$ $f^{-1}(x) = \dfrac{3x+1}{x-2}$.

5 (a) An equivalence relation on A is any relation on A which is reflexive, symmetric and transitive.

$[a] = \{x \in A : xRa\} = \{x \in A : \langle x, a\rangle \in R\}$

(b) $a^2 - a^2 = 0 = 6(0)$. $\therefore$ aRa for each $a \in \mathbb{Z}$

Suppose aRb. Then $a^2 - b^2 = 6k$ for some $k \in \mathbb{Z}$. So $b^2 - a^2 = -6k = 6(-k)$. $\therefore$ bRa. Hence for any $a, b \in \mathbb{Z}$ $aRb \rightarrow bRa$.

Finally suppose aRb and bRc. Then $a^2 - b^2 = 6k$ and $b^2 - c^2 = 6\ell$ for some $k, \ell \in \mathbb{Z}$. Thus $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2) = 6(k + \ell)$. $\therefore$ aRc.

So for any $a, b, c \in \mathbb{Z}$ $aRb \wedge bRc \rightarrow aRc$.

5 (b) Hence R is reflexive, symmetric and transitive.
The equivalence classes of R are

$[0] = \{\ldots, -6, 0, 6, 12, \ldots\}$
$[1] = \{\ldots -5, -1, 1, 5, 7, 11, \ldots\}$
$[2] = \{\ldots -4, -2, 2, 4, 8, 10, \ldots\}$
$[3] = \{\ldots -3, 0, 3, 9, 15, \ldots\}$

6(a) A function from A to B is any relation F from A to B such that $(\forall a \in A)(\exists! \ \in B)(\langle a, \ \rangle \in F)$.

(b) Suppose $f: A \to B$ and $g: B \to C$ are functions.
Existence: Let $a \in A$. Then put $b = f(a)$ and $c = g(b)$. Since $\langle a, b\rangle \in f$ and $\langle b, c\rangle \in g$, it follows that $\langle a, c\rangle \in g \circ f$. So $(\forall a \in A)(\exists c \in C) \langle a, c\rangle \in g \circ f)$

Uniqueness: Suppose $\langle a, c_1\rangle$ and $\langle a, c_2\rangle \in g \circ f$. Then $\exists b_1, b_2 \in B$ such that $\langle a, b_1\rangle \in f$ and $\langle b_1, c_1\rangle \in g$ and $\langle a, b_2\rangle \in f$ and $\langle b_2, c_2\rangle \in g$. Since f is a function and $\langle a, b_1\rangle$ and $\langle a, b_2\rangle \in f$, it follows that $b_1 = b_2$. So $\langle b_1, c_1\rangle$ and $\langle b_1, c_2\rangle \in g$. Since g is a function it follows that $c_1 = c_2$. Hence $(\forall a \in A)(\exists! c \in C)(\langle a, c\rangle \in g \circ f)$. $\therefore$ $g \circ f$ is a function from A to C.

ALT. 3(b) Assume $A \not\subseteq B$ and $A - C \subseteq B$. Suppose $A \cap C = \emptyset$. Let $a \in A$. Then $a \notin C$ because $A \cap C$. So $a \in A - C$. Since $A - C \subseteq B$ it follows that $a \in B$. But a was arbitrary. $\therefore$ $A \subseteq B$ contradicting $A \not\subseteq B$. Hence $A \cap C \neq \emptyset$.