MAA 3200 – Introd. to Adv. Math — Florida Internat'l Univ.

TEST # 2 – Fall 2004 — Time: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. Begin each question on a separate page.

(20) 1(a) Let $f: X \to Y$ be a function, $A \subseteq X$, and $C \subseteq Y$. Define what are $f[A]$ and $f^{-1}[C]$.

(b) If $A \subseteq X$ and $B \subseteq X$, prove that $f[A \cup B] = f[A] \cup f[B]$.

(15) 2. Use Mathematical Induction to prove that for each $n \geq 1$
$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

(20) 3(a) Define what is a finite set and what is a denumerable set.

(b) If $A \sim C$ and $B \sim D$, prove that $A \times B \sim C \times D$.
[If you claim that a function is bijective, you must verify this.]

(15) 4(a) Define what it means for u_0 to be the supremum (l.u.b.) of the set A of real numbers.

(b) If $A \subseteq \mathbb{R}$ and $u_0 = \text{supremum}(A)$, prove that
$(\forall \varepsilon > 0)(\exists a \in A)(u_0 - \varepsilon < a \leq u_0)$.

(15) 5(a) Define what it means for $\langle a_n \rangle_{n \geq 1}$ to be convergent.

(b) Suppose $\langle a_n \rangle$ converges to A and $\langle b_n \rangle$ converges to B. Prove that $\langle a_n - b_n \rangle$ converges to $A - B$.

(15) 6(a) Define what is a Cauchy sequence.

(b) Prove that if $\langle a_n \rangle$ is a convergent sequence, then $\langle a_n \rangle$ is a Cauchy sequence.

MAA 3200 - Introd to Adv. Math. Florida Int'l Univ.

SOLUTIONS TO TEST #2 FALL 2004

1(a) The image of A under f is defined by

$$f[A] = \{y \in Y : (\exists a \in A)(y = f(a))\}$$

The inverse image of C under f is defined by

$$f^{-1}[C] = \{x \in X : (\exists c \in C)(f(x) = c)\}$$

b) Let $y \in f[A \cup B]$. Then we can find an $a \in A \cup B$ such that $y = f(a)$. Now since $a \in A \cup B$, either $a \in A$ or $a \in B$. But if $a \in A$, then $y \in f[A]$. And if $a \in B$, then $y \in f[B]$. So either $y \in f[A]$ or $y \in f[B]$. $\therefore$ $y \in f[A] \cup f[B]$. Thus

$f[A \cup B] \subseteq f[A] \cup f[B]$ .... (1)

Now suppose $y \in f[A] \cup f[B]$. Then either $y \in f[A]$ or $y \in f[B]$. In the first case we can find an $a \in A$ such that $y = f(a)$. Since $a \in A \cup B$, $y \in f[A \cup B]$. And in the second case we can find a $b \in B$ such that $y = f(b)$. Since $b \in A \cup B$, $y \in f[A \cup B]$. So in either case $y \in f[A \cup B]$. Hence

$f[A] \cup f[B] \subseteq f[A \cup B]$ .... (2)

From (1) & (2) it follows that $f[A \cup B] = f[A] \cup f[B]$.

2. Let $P(n)$ be the statement: "$1+3+5+\cdots+(2n-1) = n^2$"

Base Case: If $n=1$, then $1+3+5+\cdots+(2n-1) = 1 = 1^2$.

So $P(1)$ is true.

Ind. step. Suppose that $n \geq 1$ and $P(n)$ is true. Then $1+3+5+\cdots+(2n-1) = n^2$. So it follows that

$$1+3+5+\cdots+(2n-1)+2(n+1)-1 = n^2+(2n+1) = (n+1)^2.$$

i.e., $P(n+1)$ will be true. Hence $P(n) \to P(n+1)$ for each $n \geq 1$.

Conclusion: By the Principle of Mathematical Induction it follows that $P(n)$ is true for each $n \geq 1$. Hence for each $n \geq 1$, $1+3+5+\cdots+(2n-1) = n^2$.

3(a) A set A is said to be finite if we can find an $n \in \mathbb{N}$ such that $A \sim \{1,2,3,\ldots,n\}$. A set A is said to be denumerable if $\mathbb{Z}^+ \sim A$. (We say that $A \sim B$ if we can find a bijection from A to B)

(b) Suppose $A \sim C$ and $B \sim D$. Then we can find bijections $f: A \to C$ and $g: B \to D$. Define $h: A \times B \to C \times D$ by $h(\langle a,b\rangle) = \langle f(a), g(b)\rangle$. We claim that h is a bijection. From this it will follow that $A \times B \sim C \times D$.

Suppose $h(\langle a_1, b_1\rangle) = h(\langle a_2, b_2\rangle)$. Then $\langle f(a_1), g(b_1)\rangle = \langle f(a_2), g(b_2)\rangle$. So $f(a_1) = f(a_2)$ and $g(b_1) = g(b_2)$. Since f and g are bijections, they are injections. So $a_1 = a_2$ and $b_1 = b_2$. Hence $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$. $\therefore$ h is injective.

Now let $\langle c,d\rangle$ be an arbitrary element of $C \times D$. Then $c \in C$ and $d \in D$. Since $f: A \to C$ and $g: B \to D$ are bijections, they are surjections. So we can find an $a \in A$ and a $b \in B$ such that $f(a) = c$ and $g(b) = d$. Hence $h(\langle a,b\rangle) = \langle f(a), g(b)\rangle = \langle c,d\rangle$. Thus h is surjective.

Since h is injective and surjective, it follows that h is a bijection. $\therefore$ $A \times B \sim C \times D$.

4. (a) We say that u_0 is the <u>supremum</u> of A if

(i) u_0 is an upper bound of A, and

(ii) $u_0 \leq u$ for all upper bounds u of A.

(b) Suppose $u_0 = \text{supremum}(A)$. Let $\varepsilon > 0$ be given. Then $u_0 - \varepsilon$ cannot be an upper bound of A, otherwise u_0 will not be the supremum of A (The supremum of A is the least upper bound of A). So we can find an $a \in A$ such that $u_0 - \varepsilon < a$. Now since u_0 is an upper bound of A, we must have $x \leq u_0$ for all $x \in A$. Hence $a \leq u_0$. So $u_0 - \varepsilon < a \leq u_0$. Thus $(\forall \varepsilon > 0)(\exists a \in A)(u_0 - \varepsilon < a \leq u_0)$.

5. (a) A sequence $\langle a_n \rangle$ is said to be convergent if $(\exists L \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N)(|a_n - L| < \varepsilon)$.
(i.e., if we can find a real number L such that for any $\varepsilon > 0$, we can find an $N \in \mathbb{Z}^+$ such that $|a_n - L| < \varepsilon$ whenever $n \geq N$.)

(b) Suppose $\langle a_n \rangle$ conv. to A and $\langle b_n \rangle$ conv. to B. Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$. Since $\langle a_n \rangle$ converges to A and $\langle b_n \rangle$ converges to B, we can find $N_1 \in \mathbb{Z}^+$ and $N_2 \in \mathbb{Z}^+$ such that

$$(\forall n \geq N_1)(|a_n - A| < \varepsilon/2) \quad \text{and}$$
$$(\forall n \geq N_2)(|b_n - B| < \varepsilon/2).$$

Let $N = \max\{N_1, N_2\}$. Then $(\forall n \geq N)$ we have

$$|(a_n - b_n) - (A - B)| = |(a_n - A) + (B - b_n)|$$
$$\leq |a_n - A| + |B - b_n|$$

5 (b) (contd.)

$$\begin{aligned} &= |a_n - A| + |b_n - B| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Hence for any $\varepsilon > 0$, we have found an $N \in \mathbb{Z}^+$ such that $(\forall n \geq N)(|(a_n - b_n) - (A - B)| < \varepsilon)$. Thus $\langle a_n - b_n \rangle$ converges to $A - B$.

6 (a) A sequence $\langle a_n \rangle$ is said to be a Cauchy sequence if $(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall m, n \geq N)(|a_n - a_m| < \varepsilon)$.

(b) Suppose $\langle a_n \rangle$ is a convergent sequence. Then we can find an $A \in \mathbb{R}$ such that $\langle a_n \rangle$ converges to A. Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$. Since $\langle a_n \rangle$ converges to A, we can find an $N \in \mathbb{Z}^+$ such that $(\forall n \geq N)(|a_n - A| < \varepsilon/2)$. So for all $m, n \geq N$ we have

$$\begin{aligned} |a_n - a_m| &= |(a_n - A) + (A - a_m)| \\ &\leq |a_n - A| + |A - a_m| \\ &= |a_n - A| + |a_m - A| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Hence for any $\varepsilon > 0$, we have found an $N \in \mathbb{Z}^+$ such that $(\forall m, n \geq N)(|a_n - a_m| < \varepsilon)$. So $\langle a_n \rangle$ is a Cauchy sequence.