MAA 3200 - INTROD TO ADV. MATH FLORIDA INT'L UNIV.

TEST #1 - FALL 2007 TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(15) 1. Translate the following argument into symbolic language and then use a truth table to determine if it is valid.
Either Amy or Cathy will pass. If Cathy passes, then Bob will not pass. $\therefore$ if Bob passes, then Amy passes.

(15) 2(a) Let A and B be sets. Define what are $A \cup B$ and $A - B$ using set building notation and connectives.
(b) Prove that $A-(B-C) = (A-B)\cup(A\cap C)$.

(15) 3(a) Show that $\neg(\forall x\in A)(\forall y\in A)[f(x)=f(y) \to (x=y)]$ is logically equivalent to $(\exists x\in A)(\exists y\in A)[(x\neq y)\wedge f(x)=f(y)]$.
(b) Prove that $(A\times B)\cap(C\times D)=\emptyset$ implies $A\cap C=\emptyset$ or $B\cap D=\emptyset$.

(15) 4(a) Let $\mathcal{F}$ be a family of sets. Define $\cup\mathcal{F}$ and $\cap\mathcal{F}$.
(b) Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of non-empty sets.
If $(\cup\mathcal{F})\cap(\cup\mathcal{G})=\emptyset$, does it follow that $\mathcal{F}\cap\mathcal{G}=\emptyset$?

(20) 5(a) Let R and S be relations on A. Define R^{-1} and $S\circ R$.
(b) Prove that $(S\circ R)^{-1} = R^{-1}\circ S^{-1}$.

(20) 6(a) Define what is an equivalence relation R on the set A.
If $a\in A$, define what is the equivalence class $[a]_R$
(b) Let R be the relation on $\mathbb{Z}$ defined by aRb if a^2-b^2 is a multiple of 8. Prove that R is an equivalence relation on $\mathbb{Z}$ and find all the equivalence classes of R.

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Solutions to Test #1 — Fall 2007.

1(a) Let A = Amy passes, B = Bob passes, and C = Cathy passes.
The argument says: $(A \vee C) \wedge (C \to \neg B) \quad \therefore \quad B \to A$.

(b)

A	B	C	$(A \vee C)$	$(C \to \neg B)$	$\wedge$	$[(A \vee C) \wedge (C \to \neg B)] \to (B \to A)$	$(B \to A)$
T	T	T	T	F	F	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	T	T	T
F	T	T	T	F	F	T	F
F	T	F	F	F	T	T	F
F	F	T	T	T	T	T	T
F	F	F	F	F	T	T	T

So the argument is valid bec. we got a tautology.

2(a) $A \cup B = \{x : x \in A \vee x \in B\}$
$A - B = \{x : x \in A \wedge x \notin B\} = \{x : (x \in A) \wedge \neg(x \in B)\}$

(b) Suppose $x \in A-(B-C)$. Then $x \in A$ and $x \notin (B-C)$. So $x \in A$, and $\neg(x \in B$ and $x \notin C)$. Thus $x \in A$, and $x \notin B$ or $x \in C$. Hence $x \in A$ and $x \notin B$, or $x \in A$ and $x \in C$ (by the distributive law). Thus $x \in A-B$ or $x \in A \cap C$. Hence $x \in (A-B) \cup (A \cap C)$
Therefore $A-(B-C) \subseteq (A-B) \cup (A \cap C)$ ... (*)
Now suppose $x \in (A-B) \cup (A \cap C)$. Then $x \in A-B$ or $x \in A \cap C$. So $x \in A$ and $x \notin B$, or $x \in A$ and $x \in C$. Hence $x \in A$, and $x \notin B$ or $x \in C$ (by the distributive law). Thus $x \in A$ and $\neg(x \in B$ and $x \notin C)$. Hence $x \in A-(B-C)$. Thus $(A-B) \cup (A \cap C) \subseteq A-(B-C)$ ... (**)
From (*) & (**), it follows that $A-(B-C) = (A-B) \cup (A \cap C)$.

3(a) $\neg(\forall x\in A)(\forall y\in A)[f(x)=f(y)\rightarrow(x=y)]$

$\Leftrightarrow (\exists x\in A)\neg(\forall y\in A)[f(x)=f(y)\rightarrow(x=y)]$ (quantifier neg. law)

$\Leftrightarrow (\exists x\in A)(\exists y\in A)\neg[\neg\{f(x)=f(y)\}\vee(x=y)]$ bec. $P\rightarrow Q\Leftrightarrow\neg P\vee Q$

$\Leftrightarrow (\exists x\in A)(\exists y\in A)[\neg\neg\{f(x)=f(y)\}\wedge\neg(x=y)]$ bec. $\neg(P\vee Q)\Leftrightarrow\neg P\wedge\neg Q$

$\Leftrightarrow (\exists x\in A)(\exists y\in A)[(x\neq y)\wedge\{f(x)=f(y)\}]$ bec. $\neg\neg P\Leftrightarrow P$ and $P\wedge Q\Leftrightarrow Q\wedge P$

(b) It will suffice to prove the contrapositive. So we will prove that $\neg(A\cap C=\emptyset\vee B\cap D=\emptyset)\Rightarrow\neg[(A\times B)\cap(C\times D)=\emptyset]$ i.e. $[A\cap C\neq\emptyset\wedge B\cap D\neq\emptyset]\rightarrow(A\times B)\cap(C\times D)\neq\emptyset$. So suppose $A\cap C\neq\emptyset$ and $B\cap D\neq\emptyset$. Then we can find an $a\in A\cap C$ and a $b\in B\cap D$. So $\langle a,b\rangle\in A\times B$ because $a\in A$ and $b\in B$, and $\langle a,b\rangle\in C\cap D$ because $a\in C$ and $b\in D$. Hence $\langle a,b\rangle\in(A\times B)\cap(C\times D)$. Thus $(A\times B)\cap(C\times D)\neq\emptyset$. $\therefore (A\times B)\cap(C\times D)=\emptyset\Rightarrow A\cap C=\emptyset$ or $B\cap D=\emptyset$.

4(a) $\cup\mathcal{F}=\{x:(\exists A\in\mathcal{F})(x\in A)\}=\{x:(\exists A)(A\in\mathcal{F}\wedge x\in A)\}$

$\cap\mathcal{F}=\{x:(\forall A\in\mathcal{F})(x\in A)\}=\{x:(\forall A)(A\in\mathcal{F}\rightarrow x\in A)\}$

(b) Yes, $(\cup\mathcal{F})\cap(\cup\mathcal{G})=\emptyset$ implies $(\mathcal{F}\cap\mathcal{G})=\emptyset$. We shall prove the contrapositive, i.e. $\mathcal{F}\cap\mathcal{G}\neq\emptyset\Rightarrow(\cup\mathcal{F})\cap(\cup\mathcal{G})\neq\emptyset$. Suppose $\mathcal{F}\cap\mathcal{G}\neq\emptyset$. Then we can find a set B such that $B\in\mathcal{F}\cap\mathcal{G}$. Since $\mathcal{F}$ and $\mathcal{G}$ consists of non-empty sets, $B\neq\emptyset$. So we can find an element $b\in B$. Now $b\in\cup\mathcal{F}$ because $b\in B$ and $B\in\mathcal{F}$; and $b\in\cup\mathcal{G}$ bec $b\in B$ and $b\in\mathcal{G}$. Hence $b\in(\cup\mathcal{F})\cap(\cup\mathcal{G})$. Thus $(\cup\mathcal{F}\cap(\cup\mathcal{G})\neq\emptyset$. So $\mathcal{F}\cap\mathcal{G}\neq\emptyset\Rightarrow(\cup\mathcal{F})\cap(\cup\mathcal{G})\neq\emptyset$. $\therefore (\cup\mathcal{F})\cap(\cup\mathcal{G})=\emptyset\Rightarrow\mathcal{F}\cap\mathcal{G}=\emptyset$.

5(a) $R^{-1}=\{\langle b,a\rangle:\langle a,b\rangle\in R\}$, $S\circ R=\{\langle a,c\rangle:(\exists b\in A)(\langle a,b\rangle\in R\wedge\langle b,c\rangle\in S)\}$

(b) Suppose $\langle x,y\rangle\in(S\circ R)^{-1}$. Then $\langle y,x\rangle\in S\circ R$. So we can find a $z\in A$ such that $\langle y,z\rangle\in R$ and $\langle z,x\rangle\in S$.

5(b) Since $\langle y,z\rangle \in R$ and $\langle z,x\rangle \in S$, $\langle x,z\rangle \in S^{-1}$ and $\langle y,z\rangle \in R^{-1}$.
So $\langle x,y\rangle \in R^{-1}\circ S^{-1}$ because $\langle x,z\rangle \in S^{-1}$ and $\langle y,z\rangle \in R^{-1}$.
$\therefore (S\circ R)^{-1} \subseteq R^{-1}\circ S^{-1}$ .... (1)
Now suppose $\langle x,y\rangle \in R^{-1}\circ S^{-1}$. Then we can find a $z\in A$ such that $\langle x,z\rangle \in S^{-1}$ and $\langle z,y\rangle \in R^{-1}$. Since $\langle x,z\rangle \in S^{-1}$ and $\langle z,y\rangle \in R^{-1}$, it follows that $\langle y,z\rangle \in R$ and $\langle z,x\rangle \in S$. So $\langle y,x\rangle \in S\circ R$. Hence $\langle x,y\rangle \in (S\circ R)^{-1}$. Thus $R^{-1}\circ S^{-1} \subseteq (S\circ R)^{-1}$ ...(2). From (1)&(2) we get $(S\circ R)^{-1} = R^{-1}\circ S^{-1}$.

6(a) An equivalence relation R on A is any relation on A that is reflexive, symmetric, and transitive. The equivalence class $[a]_R$ is defined by $[a]_R = \{x\in A : xRa\}$.

(b) For any $\in \mathbb{Z}$ we have $a^2-a^2 = 0 = 8(0)$. So a^2-a^2 is a multiple of 8. $\therefore$ aRa for each $a\in\mathbb{Z}$. Hence R is reflexive.
Now suppose aRb. Then $a^2-b^2 = 8k$ for some $k\in\mathbb{Z}$. $\therefore$ $b^2-a^2 = -(a^2-b^2) = -8k = 8(-k)$. Hence bRa. So $(\forall a\in\mathbb{Z})(\forall b\in\mathbb{Z})(aRb \to bRa)$, Thus R is symmetric.
Finally, suppose aRb and bRc. Then $a^2-b^2=8k$ and $b^2-c^2 = 8\ell$ for some integers $k,\ell\in\mathbb{Z}$. Hence $a^2-c^2 = (a^2-b^2)+(b^2-c^2) = 8k+8k = 8(k+\ell)$. Thus aRb and bRc implies aRc. So $(\forall a\in\mathbb{Z})(\forall b\in\mathbb{Z})(\forall c\in\mathbb{Z})(aRb \wedge bRc \to aRc)$. Thus R is transitive.
Hence R is an equivalence relation.

(c) The equivalence classes are

$[0]_R = \{x\in\mathbb{Z} : x^2-0^2 = 8k, k\in\mathbb{Z}\} = \{8k : k\in\mathbb{Z}\} \cup \{8k+4 : k\in\mathbb{Z}\}$

$[1]_R = \{x\in\mathbb{Z} : x^2-1^2 = 8k, k\in\mathbb{Z}\} = \{8k\pm1 : k\in\mathbb{Z}\} \cup \{8k\pm3 : k\in\mathbb{Z}\}$

$[2]_R = \{x\in\mathbb{Z} : x^2-2^2 = 8k, k\in\mathbb{Z}\} = \{8k\pm2 : k\in\mathbb{Z}\}$