MAA 3200 - Introd. to Adv. Math — Florida Int'l Univ.

TEST # 2 - Fall 2007 — TIME : 75 min.

Answer ALL 6 questions. An unjustified answer will receive little credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(20) 1(a) Let $f : A \to B$ be a function. Define what it means for $f$ to be injective & what it means for $f$ to be surjective.

(b) Let $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$ be defined by $f(x) = \frac{x}{x-3}$.
(i) Is $f$ injective? (ii) Is $f$ surjective?

(15) 2. Let $f : X \to Y$ be a function and suppose $A, B \subseteq X$.

(a) Is it always true that $f[A \cap B] \subseteq f[A] \cap f[B]$?

(b) Is it always true that $f[A] \cap f[B] \subseteq f[A \cap B]$?

(15) 3(a) Write down the First Principle of Mathematical Induction

(b) Prove that $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all $n \geq 0$.

(15) 4(a) Define what it means for the set $A$ to be finite and what it means for $A$ to be denumerable.

(b) Prove that $\mathbb{Z}$ is a denumerable set.

(15) 5(a) Define what it means for $\ell_0$ to be the infimum (g.l.b) of $A$.

(b) If $A \subseteq \mathbb{R}$ and $\ell_0$ is the infimum of $A$, prove that $(\forall \varepsilon > 0)(\exists a \in A)(\ell_0 \leq a < \ell_0 + \varepsilon)$

(20) 6(a) Define what it means for $\langle a_n \rangle_{n \geq 1}$ to be convergent.

(b) If $\langle a_n \rangle_{n \geq 1}$ is convergent, prove that $(\exists M \in \mathbb{R}^+)(\forall n \in \mathbb{Z}^+)(a_n \leq M)$.

(c) If $\langle a_n \rangle_{n \geq 1}$ converges to $A$ and $\langle b_n \rangle_{n \geq 1}$ converges to $B$, prove that $\langle a_n + b_n \rangle$ converges to $A + B$.

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Solutions to Test #2 — Fall 2007.

1(a) $f$ is injective if $(\forall x_1 \in A)(\forall x_2 \in A)\,[f(x_1) = f(x_2) \to x_1 = x_2]$

$f$ is surjective if $(\forall y \in B)(\exists x \in A)\,[f(x) = y]$.

(b)(i) Suppose $f(x_1) = f(x_2)$. Then $\dfrac{x_1}{x_1 - 3} = \dfrac{x_2}{x_2 - 3}$.

$\therefore\ x_1(x_2 - 3) = x_2(x_1 - 3)$

$\therefore\ x_1 x_2 - 3x_1 = x_2 x_1 - 3x_2 \Rightarrow -3x_1 = -3x_2$

$\therefore\ x_1 = x_2$. So $f$ is injective.

(ii) Take any $y \in \mathbb{R} - \{1\}$. Choose $x = 3y/(y-1)$. Since $x = 3/(1 - 1/y)$, $x \neq 3$. So $x \in \mathbb{R} - \{3\}$. Also

$$f(x) = \frac{x}{x-3} = \frac{3y/(y-1)}{[3y/(y-1)] - 3} = \frac{(3y)/(y-1)}{[3y - 3(y-1)]/(y-1)}$$

$$= \frac{3y}{y-1} \cdot \frac{y-1}{3y - (3y-3)} = \frac{3y}{3} = y$$

So for any $y \in \mathbb{R} - \{1\}$ we have found an $x \in \mathbb{R} - \{3\}$ with $f(x) = y$. Hence $f$ is surjective

[$x$ was obtained by solving $f(x) = y$ for $x$ in terms of $y$.]

2(a) If $C \subseteq X$, then $f[C] = \{f(a) : a \in C\}$ by the definition. Now let $y \in f[A \cap B]$. Then $y = f(a)$ for some $a \in A \cap B$. Since $a \in A \cap B$, $a \in A$; so $y = f(a)$ with $a \in A$. Hence $y \in f[A]$. Also since $a \in A \cap B$, $a \in B$; so $y = f(a)$ with $a \in B$. Hence $y \in f[B]$. Thus $y \in f[A] \cap f[B]$. So $f[A \cap B] \subseteq f[A] \cap f[B]$ is always true.

(b) Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(k) = k^2$. Also let $A = \{0, 3\}$ and $B = \{0, -3\}$. Then $A \cap B = \{0\}$

$f[A] \cap f[B] = \{0, 9\} \not\subseteq \{0\} = f[A \cap B]$. So $f[A] \cap f[B]$ is not always a subset of $f[A \cap B]$.

3(a) Let $P(n)$ be a statement for each $n \in \mathbb{N}$. Suppose $P(0)$ is true and $(\forall n \in \mathbb{N})[P(n) \to P(n+1)]$ is true. Then $(\forall n \in \mathbb{N})[P(n)]$ is true.

(b) Let $P(n)$ be the statement "$2^{n+2} + 3^{2n+1}$ is divisible by 7." Since $2^{0+2} + 3^{2(0)+1} = 4 + 3 = 7$, $P(0)$ is true. Now suppose $P(n)$ is true. Then $2^{n+2} + 3^{2n+1}$ is divisible by 7. Now

$$2^{(n+1)+2} + 3^{2(n+1)+1} = 2 \cdot 2^{n+2} + 3^2 \cdot 3^{2n+1}$$
$$= 2 \cdot \underbrace{\left(2^{n+2} + 3^{2n+1}\right)}_{\text{divisible by 7}} + \underbrace{7 \cdot 3^{2n+1}}_{\text{divisible by 7}}$$

So $P(n+1)$ will be true. $\therefore (\forall n \in \mathbb{N})[P(n) \to P(n+1)]$ is true. $\therefore$ by the P.M.I. $(\forall n \in \mathbb{N})\, P(n)$ is true. Hence $2^{n+2} + 3^{2n+1}$ is divisible by 7 for all $n \geq 0$.

4(a) The set $A$ is finite if we can find a bijection $f: \{1, 2, 3, \ldots, n\} \to A$ for some $n \in \mathbb{N}$. $A$ is denumerable if we can find a bijection $f: \mathbb{Z}^+ \to A$.

(b) Let $f: \mathbb{Z}^+ \to \mathbb{Z}$ be defined by $f(k) = \begin{cases} k/2 & \text{if } k \in \mathbb{E}^+ \\ (1-k)/2 & \text{if } k \in \mathbb{O}^+ \end{cases}$

Here $\mathbb{E}^+$ = set of even positive integers & $\mathbb{O}^+$ = set of odd positive integers. It is clear that $f$ is injective on $E^+$ and on $\mathbb{O}^+$ because $k/2$ and $(1-k)/2$ are linear functions. Also $f[\mathbb{E}^+] = \mathbb{Z}^+$ and $f[\mathbb{O}^+] = \mathbb{Z} - \mathbb{Z}^+$. Since $f[\mathbb{E}^+]$ & $f[\mathbb{O}^+]$ are disjoint, $f$ is injective. Also since $f[\mathbb{E}^+] \cup f[\mathbb{O}^+] = \mathbb{Z}^+ \cup (\mathbb{Z} - \mathbb{Z}^+) = \mathbb{Z}$, $f$ is surjective. Thus $f$ is a bijection.

5(a) $l_0$ is the infimum (greatest lower bound) of $A$ if
(i) $l_0$ is a lower bound of $A$, and
(ii) $l_0 \geq l$ for any lower bound of $A$.

5(a) Let $\varepsilon > 0$ be given. Then $l_0 < l_0 + \varepsilon$. Since $l_0$ was the greatest lower bound of $A$, $l_0 + \varepsilon$ cannot be a lower bound of $A$. So we must be able to find an $a \in A$ with $a < l_0 + \varepsilon$. Also, since $l_0$ is a lower bound of $A$, $(\forall x \in A)(l_0 \leq x)$. So, in particular $a \leq l_0$. Hence $l_0 \leq a < l_0 + \varepsilon$. Since $\varepsilon$ was arbitrary, this means that

$$(\forall \varepsilon > 0)(\exists a \in A)[l_0 \leq a < l_0 + \varepsilon] .$$

6(a) The sequence $\langle a_n \rangle_{n \geq 1}$ is convergent if

$$(\exists L \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N)(|a_n - L| < \varepsilon)$$

(b) Suppose $\langle a_n \rangle_{n \geq 1}$ is convergent. Then $(\exists L \in \mathbb{R})$ $(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N)(|a_n - L| < \varepsilon)$. Take $\varepsilon = 1$. Then we can find an $N \in \mathbb{Z}^+$ such that $(\forall n \geq N)(|a_n - L| < 1)$. So $\forall n \geq N$ we have

$$-1 < a_n - L < 1, \quad \text{i.e.} \quad L - 1 < a_n < L + 1 \leq |L| + 1$$

Let $M = \max\{a_1, a_2, a_3, \ldots, a_{N-1}, |L| + 1\}$. Then $(\forall n \in \mathbb{Z}^+)(a_n \leq M)$ and $M \in \mathbb{R}^+$.

(c) Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$. Since $\langle a_n \rangle_{n \geq 1}$ converges to $A$ we can find an $N_1 \in \mathbb{Z}^+$ such that $(\forall n \geq N_1)(|a_n - A| < \varepsilon/2)$. Also since $\langle b_n \rangle_{n \geq 1}$ converges to $B$, we can find an $N_2 \in \mathbb{Z}^+$ such that $(\forall n \geq N_2)(|a_n - B| < \varepsilon/2)$. Let $N = \max\{N_1, N_2\}$. Then $(\forall n \geq N)$ we have

$$\begin{aligned} |(a_n + b_n) - (A + B)| &= |(a_n - A) + (b_n - B)| \\ &\leq |a_n - A| + |b_n - B| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

So $(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N)(|(a_n + b_n) - (A + B)| < \varepsilon)$

$\therefore$ $\langle a_n + b_n \rangle_{n \geq 1}$ converges to $A + B$.