MAD 3200 - Introd to Adv. Math — Florida Int'l Univ.

Test #1 - Fall 2008 — TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(15) 1 (a) Translate the following argument into symbolic language

(b) Then use a truth table to determine if it is valid.

If Peter cheated, then Qasim cheated. Either Peter or Raul cheated. $\therefore$ if Qasim did not cheat, then Raul cheated.

(15) 2 (a) Define $(\exists x \in A)P(x)$ and $(\forall x \in A)P(x)$ in terms of unbounded Quantifiers.

(b) Convert $\neg(\exists x)(\forall y)[f(x,y)=0 \rightarrow y=0]$ into an equivalent formula in which no "$\neg$" governs a quantifier or connective.

(15) 3 (a) If $\mathcal{F}$ and $\mathcal{G}$ are families of sets, define $\mathcal{F} \cup \mathcal{G}$ and $\cap \mathcal{F}$.

(b) Let A, B, and C be sets. If $A - B \subseteq C$, does it follow that $A - C \subseteq B$. (Give a proof or provide a counter-example.)

(15) 4 (a) Let R be a relation from A to B and S be a relation from B to C. Define S^{-1} and $S \circ R$.

(b) Let R be an equiv. relation and $b \in [a]$. Prove that $[b] = [a]$.

(20) 5 (a) Define what is a function from A to B.

(b) Let R be the relation on $\mathbb{Z}$ defined by aRb if $a^2 - b^2$ is a multiple of 9. Prove that R is an equivalence relation.

(c) Find all the equivalence classes of R

(20) 6 (a) Let $\langle A_i : i \in I \rangle$ be an indexed family of subsets of $\mathcal{U}$. Define $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$

(b) Prove that $\mathcal{U} - \left(\bigcap_{i \in I} A_i\right) = \bigcup_{i \in I} (\mathcal{U} - A_i)$

MAA 3200 - Introd. to Adv. Math — Florida Int'l Univ.

Solutions to Test #1 — Fall 2008

1 (a) Let P = Peter cheated, Q = Qasim cheated & R = Raul cheated

Argument says: $(P \to Q) \wedge (P \vee R) \quad \therefore \quad (\neg Q \to R)$

(b) Argument will be valid if $[(P \to Q) \wedge (P \vee R)] \to (\neg Q \to R)$ is a tautology. Below it is shown to be valid.

P	Q	R	$[(P \to Q)$	$\wedge$	$(P \vee R)]$	$\to$	$[(\neg Q) \to R]$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	F	F	T	F

2 (a) $(\exists x \in A)\, P(x)$ means $(\exists x)[x \in A \wedge P(x)]$

$(\forall x \in A)\, P(x)$ means $(\forall x)[x \in A \to P(x)]$

(b) $\neg (\exists x)(\forall y)[f(x,y) = 0 \to y = 0]$

$\Leftrightarrow (\forall x) \neg (\forall y)[\neg\{f(x,y) = 0\} \vee (y = 0)]$ Quant. neg. Rule & Rewriting "$\to$"

$\Leftrightarrow (\forall x)(\exists y) \neg [\neg\{f(x,y) = 0\} \vee (y = 0)]$ Quant. neg. rule

$\Leftrightarrow (\forall x)(\exists y)[\neg\neg\{f(x,y) = 0\} \wedge \neg (y = 0)]$ De Morgan's R.

$\Leftrightarrow (\forall x)(\exists y)[f(x,y) = 0 \wedge \neg (y = 0)]$ double neg. R.

3 (a) $\mathcal{F} \cup \mathcal{G} = \{A : A \in \mathcal{F} \vee A \in \mathcal{G}\}$, $\quad \cap \mathcal{F} = \{x : (\forall A \in \mathcal{F})(x \in A)\}$.

(b) Assume $A - B \subseteq C$. Let $x \in A - C$. Then $x \in A$ and $x \notin C$. Now suppose $x \notin B$. Then $x \in A$ and $x \notin B$. So $x \in A - B$. Since $A - B \subseteq C$, it follows that $x \in C$. But this contradicts the fact that $x \notin C$. So we must have $x \in B$. $\therefore A - C \subseteq B$. Hence if $A - B \subseteq C$, then it follows that $A - C \subseteq B$.

4(a) $S^{-1} = \{\langle c,b\rangle : \langle b,c\rangle \in S\}$, $S \circ R = \{\langle a,c\rangle : (\exists b \in B)(\langle a,b\rangle \in R \wedge \langle b,c\rangle \in S\}$.

(b) Suppose $x \in [b]$. Then xRb. Also since $b \in [a]$, bRa. So xRb and bRa. $\therefore$ xRa because R is transitive $\therefore$ $x \in [a]$. Hence $[b] \subseteq [a]$.
Now suppose $x \in [a]$. Then xRa. Also since $b \in [a]$, bRa; and so aRb because R is symmetric. Hence xRa and aRb. $\therefore$ xRb because R is transitive So $x \in [b]$. Hence $[a] \subseteq [b]$. Thus $[a] = [b]$.

5 (a) A function from A to B is any relation F from A to B such that $(\forall a \in A)(\exists! b \in B)(\langle a,b\rangle \in F)$.

(b) For any $a \in \mathbb{Z}$, we have $a^2 - a^2 = 0 = 9(0)$. So $a^2 - a^2$ is a multiple of 9. $\therefore$ $(\forall a \in \mathbb{Z})(aRa)$. So R is reflexive.
Now suppose aRb. Then $a^2 - b^2 = 9k$ for some $k \in \mathbb{Z}$. So $b^2 - a^2 = -(a^2 - b^2) = -9k = 9(-k)$. Hence bRa. So $(\forall a \in \mathbb{Z})(\forall b \in \mathbb{Z})(aRb \to bRa)$. $\therefore$ R is symmetric.
Finally suppose aRb and bRc. Then $a^2 - b^2 = 9k$ and $b^2 - c^2 = 9\ell$ for some integers $k, \ell \in \mathbb{Z}$. So $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2) = 9k + 9\ell = 9(k+\ell)$. Hence aRc. So $(\forall a,b,c \in \mathbb{Z})(aRb \wedge bRc \to aRc)$. $\therefore$ R is transitive.
Hence R is an equivalence relation.

(c) The equivalence classes can be found by looking at the sets $C_a = \{9k + a : k \in \mathbb{Z}\}$ for $a \in \{0,1,2,3,4,5,6,7,8\}$.
The equivalence classes are the following four.

$[0]_R = \{x \in \mathbb{Z} : x^2 - 0^2 = 9k,\ k \in \mathbb{Z}\} = \{9k : k \in \mathbb{Z}\} \cup \{9k \pm 3 : k \in \mathbb{Z}\} = \{3k : k \in \mathbb{Z}\}$

$[1]_R = \{x \in \mathbb{Z} : x^2 - 1^2 = 9k,\ k \in \mathbb{Z}\} = \{9k \pm 1 : k \in \mathbb{Z}\}$

$[2]_R = \{x \in \mathbb{Z} : x^2 - 2^2 = 9k,\ k \in \mathbb{Z}\} = \{9k \pm 2 : k \in \mathbb{Z}\}$

$[4]_R = \{x \in \mathbb{Z} : x^2 - 4^2 = 9k,\ k \in \mathbb{Z}\} = \{9k \pm 4 : k \in \mathbb{Z}\}$.

Notice that $[0]_R = C_0 \cup C_3 \cup C_6$, $[1]_R = C_1 \cup C_8$, $[2]_R = C_2 \cup C_7$ and $[4]_R = C_4 \cup C_5$.

6(a) $\bigcup_{i\in I} A_i = \{x : (\exists i\in I)(x\in A_i)\}$, $\bigcap_{i\in I} A_i = \{x : (\forall i\in I)(x\in A_i)\}$

(b) Suppose $x\in \mathcal{U}-(\bigcap_{i\in I} A_i)$. Then $x\in\mathcal{U}$ and $x\notin \bigcap_{i\in I} A_i$.
So $x\in\mathcal{U}$ and $x\notin A_i$ for at least one $i\in I$. Hence
$x\in \mathcal{U}-A_i$ for at least one $i\in I$. $\therefore$ $x\in \bigcup_{i\in I}(\mathcal{U}-A_i)$.
Thus $\mathcal{U}-(\bigcap_{i\in I} A_i) \subseteq \bigcup_{i\in I}(\mathcal{U}-A_i)$ . . . (*)
Now suppose $x\in \bigcup_{i\in I}(\mathcal{U}-A_i)$. Then $x\in\mathcal{U}-A_i$ for at least one $i\in I$. So $x\in\mathcal{U}$ and $x\notin A_i$ for at least one $i\in I$. Hence $x\in\mathcal{U}$ and $x\notin \bigcap_{i\in I} A_i$. Thus $x\in \mathcal{U}-(\bigcap_{i\in I} A_i)$. $\therefore$ $\bigcup_{i\in I}(\mathcal{U}-A_i) \subseteq \mathcal{U}-(\bigcap_{i\in I} A_i)$ . . . (**)
From (*) & (**), it follows that $\mathcal{U}-(\bigcap_{i\in I} A_i) = \bigcup_{i\in I}(\mathcal{U}-A_i)$.

Below are some alternative answers.

2(b) $(\forall x)(\exists x)[f(x,y)=0 \ \wedge\ y\neq 0]$

3(a) $\mathcal{F}\cup\mathcal{G} = \{B : B\in\mathcal{F} \text{ or } B\in\mathcal{G}\}$, $\cap\mathcal{F} = \{x : (\forall A)(A\in\mathcal{F}\to x\in A)\}$
$= \{x : x \text{ is in every set } A \text{ that is a member of } \mathcal{F}\}$

3(b)
$$\begin{aligned}
A-B\subseteq C &\Rightarrow (\forall x)[(x\in A\wedge x\notin B)\to x\in C]\\
&\Rightarrow (\forall x)[\neg(x\in A\wedge x\notin B)\vee x\in C]\\
&\Rightarrow (\forall x)[x\in A\vee x\in B\vee x\in C]\\
&\Rightarrow (\forall x)[x\in A\vee x\in C\vee x\in B]\\
&\Rightarrow (\forall x)[\neg(x\in A\wedge x\notin C)\vee x\in B]\\
&\Rightarrow (\forall x)[(x\in A\wedge x\notin C)\to x\in B] \Rightarrow A-C\subseteq B.
\end{aligned}$$

5(a) A function from A to B is a relation F from A to B with $\mathrm{dom}(F)=A$ such that $(\forall x\in A)(\forall y,z\in B)[xFy\wedge xFz\to y=z]$

6(a) $\bigcup_{i\in I} A_i = \{x : x\in A_i \text{ for at least one } i\in I\}$
$\bigcap_{i\in I} A_i = \{x : x\in A_i \text{ for every } i\in I\}$