MAA 3200 - Introd to Adv Math Florida Int'l Univ.

Test #2 - Fall 2008 TIME: 75 min.

Answer all 6 questions. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(20) 1(a) Let $f: A \to B$ be a function. Define what it means for f to be injective & what it means for f to be surjective.

(b) Let $f: \mathbb{R}-\{2\} \to \mathbb{R}-\{3\}$ be defined by $f(x) = 3x/(x-2)$.

(i) Is f injective? (ii) Is f surjective?

(15) 2. Let $f: A \to B$ be a function. Prove that

(a) if $(\exists g: B \to A)(g \circ f = i_A)$ then f is injective

(b) if $(\exists h: B \to A)(f \circ h = i_B)$ then f is surjective

(15) 3(a) Define what it means for $\langle a_n \rangle_{n \geq 1}$ to be convergent.

(b) Prove that $(\forall n \in \mathbb{N})(6^n - 5n - 1$ is divisible by $25)$ by using Mathematical Induction.

(15) 4(a) Define what is a finite set & what is a denumerable set.

(b) Let $E = \{k \in \mathbb{Z} : k \text{ is even}\}$. Prove that E is denumerable.

(15) 5(a) Define what it means for u_0 to be the supremum (l.u.b) of A.

(b) If A is a non-empty subset of $\mathbb{R}$ and $u_0 = \sup(A)$, prove that $(\forall \varepsilon > 0)(\exists a \in A)(u_0 - \varepsilon < a \leq u_0)$.

(20) 6(a) Let $f: X \to Y$ be a function; $A, B \subseteq X$ and $C \subseteq Y$. Define what are $f[A]$ and $f^{-1}[C]$.

(b) Is it always true that $f[A] - f[B] \subseteq f[A-B]$?

(c) Is it always true that $f[A-B] \subseteq f[A] - f[B]$?

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Solutions to Test #2 — Fall 2008

1(a) f is injective if $(\forall x_1 \in A)(\forall x_2 \in A)\,[f(x_1) = f(x_2) \rightarrow x_1 = x_2]$.
f is surjective if $(\forall y \in B)(\exists x \in A)\,[f(x) = b]$

(b) (i) Suppose $f(x_1) = f(x_2)$. Then $3x_1/(x_1-2) = 3x_2/(x_2-2)$
So $3x_1(x_2-2) = 3x_2(x_1-2)$. $\therefore$ $3x_1x_2 - 6x_1 = 3x_1x_2 - 6x_2$
$\therefore$ $-6x_1 = -6x_2 \Rightarrow x_1 = x_2$. $\therefore$ f is injective.

(ii) Take any $y \in \mathbb{R} - \{3\}$. Choose $x = 2y/(y-3)$. Then
$x = 2/(1-3/y)$ is never 2, since $3/y$ is never 0.
So $x \in \mathbb{R} - \{2\}$. Also x is well-defined bec. $y \neq 3$ and

$$f(x) = \frac{3x}{x-2} = \frac{3 \cdot 2y/(y-3)}{2y/(y-3) - 2} = \frac{6y/(y-3)}{(2y-2y+6)/(y-3)}$$

$$= \frac{6y}{y-3} \cdot \frac{y-3}{6} = \frac{6y}{6} = y.$$

Hence f is surjective.

2(a) Assume $g: B \to A$ is a function such that $g \circ f = i_A$.
Now suppose $f(x_1) = f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$
So $(g \circ f)(x_1) = (g \circ f)(x_2)$. $\therefore$ $i_A(x_1) = i_A(x_2)$.
Hence $x_1 = x_2$. So f is injective.

(b) Assume $h: B \to A$ is a function such that $f \circ h = i_B$.
Take any $b \in B$. Choose $a = h(b)$. Then

$$f(a) = f(h(b)) = (f \circ h)(b) = i_B(b) = b.$$

So f is surjective

3(a) $\langle a_n \rangle_{n \geq 1}$ is convergent if $(\exists L \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N)$
$(|a_n - L| < \varepsilon)$

3(b) Let $P(n)$ be the statement "$6^n - 5n - 1$ is divisible by 25".
Since $6^0 - 5(0) - 1 = 0$ is divisible by 25, it follows that $P(0)$ is true. Now suppose $P(n)$ is true. Then $6^n - 5n - 1$ will be divisible by 25. Now

$$6^{n+1} - 5(n+1) - 1 = 6 \cdot \underbrace{(6^n - 5n - 1)}_{\text{div. by } 25} + \underbrace{25n}_{\text{div. by } 25}$$

So $6^{n+1} - 5(n+1) - 1$ will be divisible by 25, i.e., $P(n+1)$ will be true. $\therefore (\forall n \in \mathbb{N})[P(n) \to P(n+1)]$.
By the Princ. of Math. Ind. we get $(\forall n \in \mathbb{N})\, P(n)$. Hence $6^n - 5n - 1$ is divisible by 25 for all $n \in \mathbb{N}$.

4(a) The set S is finite if we can find a bijection $f: \{1, 2, 3, \cdots, n\} \to S$ for some $n \in \mathbb{N}$. S is denumerable if we can find a bijection $f: \mathbb{Z}^+ \to S$.

(b) Let $f: \mathbb{Z}^+ \to E$ be defined by $f(x) = x$, if x is even; and $f(x) = -(x-1)$ if x is odd. Then $f: \mathbb{Z}^+ \cap E \to E \cap \mathbb{Z}^+$ is clearly a bijection and $f: \mathbb{Z}^+ \cap \text{Odd} \to E \cap (\mathbb{Z} - \mathbb{Z}^+)$ is also a bijection. Since the domains are disjoint and the ranges are disjoint, the union of these two parts of f will produce a bijection from $\mathbb{Z}^+$ to E. Hence E is denumerable.

5(a) u_0 is the supremum of A if u_0 is an upper bound of A, and $u_0 \le u$ for any upper bound of A.

(b) Let $\varepsilon > 0$ be given. Then $u_0 - \varepsilon < u_0$. Since u_0 was the least upper bound of A, $u_0 - \varepsilon$ cannot be an upper bound of A. So $(\exists a \in A)(u_0 - \varepsilon < a)$. Also since u_0 was an upper bound of A $(\forall x \in A)(x \le u_0)$. So, in particular, $a \le u_0$. Hence $u_0 - \varepsilon < a \le u_0$. Since $\varepsilon > 0$ was arb., it follows that $(\forall \varepsilon > 0)(\exists a \in A)(u_0 - \varepsilon < a \le u_0)$.

6(a) $f[A] = \{f(a) : a \in A\}$, $\quad f^{-1}[C] = \{a \in X : f(a) \in C\}$

(b) YES. Let $y \in f[A] - f[B]$. Then $y \in f[A]$ and $y \notin f[B]$. So we can find an $a \in A$ such that $y = f(a)$ and $(\forall b \in B)[y \neq f(b)]$. Since $y = f(a)$ and $(\forall b \in B)[y \neq f(b)]$, a cannot be in B. So $a \in A - B$. Hence $f(a) \in f[A-B]$. But $y = f(a)$. So $y \in f[A-B]$. Since y was an arbitrary element of $f[A] - f[B]$, it follows that $f[A] - f[B] \subseteq f[A-B]$.

(c) NO. Let $f : \{-2, 0, 2\} \to \{0, 4\}$ be defined by $f(x) = x^2$. Take $A = \{-2, 0\}$ and $B = \{2\}$. Then

$$f[A-B] = f[\{-2,0\}] = \{f(-2), f(0)\} = \{0,4\}$$

$$\& \quad f[A] - f[B] = f[\{-2,0\}] - f[\{2\}] = \{f(-2), f(0)\} - \{f(2)\}$$
$$= \{0,4\} - \{4\} = \{0\} \neq \{0,4\} = f[A-B].$$

So $f[A-B] \nsubseteq f[A] - f[B]$ in general.

Additional Comments & alternatives answers

1(b) (ii) We had to show $(\forall y \in \mathbb{R} - \{3\})(\exists x \in \mathbb{R} - \{2\})(f(x) = y)$. We found x by solving the equation $f(x) = y$ for x in terms of y. $3x/(x-2) = y \Rightarrow 3x = xy - 2y \Rightarrow 2y = x(y-3) \Rightarrow x = 2y/(y-3)$. We had to check that $x \neq 2$ because x was supposed to be a member of $\mathbb{R} - \{2\}$. Note also that since $y \neq 3$, x was well-defined.

4(b) We can find other bijections from $\mathbb{Z}^+$ to E. Let $f_1 : \mathbb{Z} \cap E \to E$, $f_1(x) = -x$ and $f_2 : \mathbb{Z} \cap \text{Odd} \to E$, $f_2(x) = x - 1$. If we put $f = f_1 \cup f_2$, then f will be a bijection from $\mathbb{Z}^+$ to E.

6(a) $f[A] = \{b \in Y : (\exists a \in A)(f(a) = b)\}$ $\qquad f^{-1}[C] = \{a \in X : (\exists b \in C)(f(a) = b)\}$