

Answer all 6 questions. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

(15) 1(a) Translate the following argument into symbolic language.

If Ada is rich, then Beth will not be rich. Either Ada or Cindy is rich. \therefore if Beth is rich, then Cindy is rich.

(b) Use a truth table to determine if the argument is valid.

(15) 2(a) Define $(\exists x \in B) Q(x)$ & $(\forall x \in B) Q(x)$ in terms of unbounded quantifiers

(b) Convert $\neg(\forall x)(\exists z)[R(x, z) \rightarrow (x=1 \vee z \neq 0)]$ into an equivalent formula in which no " \neg " governs a quantifier or a connective.

(15) 3(a) Let \mathcal{F} and \mathcal{G} be families of sets. Define $\cup \mathcal{F}$ and $\cap \mathcal{G}$ by using unbounded quantifiers

(b) Prove that $A - (C - B) = (A \cap B) \cup (A - C)$.

(20) 4(a) Let R and S be relations on a set A . Define what is $S \circ R$.

(b) Let R be the relation on \mathbb{Z} defined by aRb if $a^2 - b^2$ is an integer multiple of 12. Prove that R is an equivalence relation and find all the equivalence classes of R .

(20) 5(a) Define what it means when we say that the set F is a function.

(b) Let $f: R - \{4\} \rightarrow R - \{3\}$ be the function define by $f(x) = (3x+1)/(x-4)$. Prove that f is a bijection.

(15) 6(a) Let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be indexed families of sets.

Define $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} B_i$.

(b) Is it always true that $\bigcup_{i \in I} (A_i \cap B_i) = (\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i)$?

1(a) Let $A = \text{Ada is rich}$, $B = \text{Beth is rich}$, & $C = \text{Cindy is rich}$.

The argument says: $(A \rightarrow \neg B) \wedge (A \vee C) \therefore (B \rightarrow C)$.

(b) The argument is valid because $(A \rightarrow \neg B) \wedge (A \vee C) \rightarrow (B \rightarrow C)$ is a tautology.

A	B	C	$(A \rightarrow \neg B) \wedge (A \vee C)$	\rightarrow	$(B \rightarrow C)$
T	T	T	F F F T	T	T
T	T	F	F F F T	T	F
T	F	T	T T T T	T	T
T	F	F	T T T T	T	T
F	T	T	T F T T	T	T
F	T	F	T F F F	T	F
F	F	T	T T T T	T	T
F	F	F	T T F F	T	T

2(a) $(\exists x \in B)Q(x)$ means $(\exists x)[x \in B \wedge Q(x)]$, $(\forall x \in B)Q(x)$ means $(\forall x)[x \in B \rightarrow Q(x)]$

$$\begin{aligned}
 (b) \quad & \neg(\forall x)(\exists z)[R(x, z) \rightarrow (x=1 \vee z \neq 0)] && \text{Quantifier neg. rule} \\
 & \Leftrightarrow (\exists x) \neg(\exists z)[\neg R(x, z) \vee (x=1 \vee z \neq 0)] && \neg P \rightarrow Q \Leftrightarrow \neg P \vee Q \\
 & \Leftrightarrow (\exists x)(\forall z)\neg[\neg R(x, z) \vee (x=1 \vee z \neq 0)] && \exists \text{ quantifier neg. rule} \\
 & \Leftrightarrow (\exists x)(\forall z)[\neg\neg R(x, z) \wedge \neg(x=1 \vee z \neq 0)] && \text{de Morgan's rule} \\
 & \Leftrightarrow (\exists x)(\forall z)[R(x, z) \wedge (\neg(x=1) \wedge \neg(z \neq 0))] && \text{Double negation rule} \\
 & \Leftrightarrow (\exists x)(\forall z)[R(x, z) \wedge x \neq 1 \wedge z = 0] && \text{de Morgan's rule} \\
 & && \text{Double negation rule.}
 \end{aligned}$$

3(a) $U\mathcal{I} = \{x : (\exists A)(A \in \mathcal{I} \wedge x \in A)\}$, $\cap\mathcal{I} = \{x : (\forall A)(A \in \mathcal{I} \rightarrow x \in A)\}$.

(b) Suppose $x \in A - (C-B)$. Then $x \in A$ and $x \notin (C-B)$. So $x \in A$ and $\neg(x \in C \wedge x \notin B)$. Thus $x \in A \wedge (x \notin C \vee x \in B)$. Hence $(x \in A \wedge x \notin C) \vee (x \in A \wedge x \in B)$. $\therefore x \in A \cap B$ or $x \in A - C$

So $x \in (A \cap B) \cup (A - C)$. Thus $A - (C-B) \subseteq (A \cap B) \cup (A - C)$... (*)

Now suppose $x \in (A \cap B) \cup (A - C)$. Then $x \in A \cap B$ or $x \in A - C$.

So $(x \in A \wedge x \in B) \vee (x \in A \wedge x \notin C)$. $\therefore x \in A \wedge (x \in B \vee x \notin C)$

$\therefore x \in A \wedge \neg(x \in C \wedge x \notin B)$. $\therefore x \in A - (C-B)$. Thus

$(A \cap B) \cup (A - C) \subseteq A - (C-B)$... (**). $\therefore A - (C-B) = (A \cap B) \cup (A - C)$.

- 4(a) $S \circ R = \{(a, c) : (\exists b \in A)(\langle a, b \rangle \in R \text{ and } \langle b, c \rangle \in S)\}$.
- (b) $a^2 - a^2 = 0 = 12(0)$. $\therefore (\forall a \in \mathbb{Z}) aRa$. So R is reflexive.
- Supp. aRb . Then $(\exists k \in \mathbb{Z})(a^2 - b^2 = 12k)$. So $b^2 - a^2 = 12(-k)$.
- $\therefore bRa$. So $(\forall a, b \in \mathbb{Z})(aRb \rightarrow bRa)$. $\therefore R$ is symmetric
- Finally supp. $aRb \wedge bRc$. Then $(\exists k, l \in \mathbb{Z}) \{a^2 - b^2 = 12k$
 $\text{and } b^2 - c^2 = 12l\}$. So $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2) = 12(k+l)$.
- $\therefore aRc$. So $(\forall a, b, c \in \mathbb{Z})(aRb \wedge bRc \rightarrow aRc)$. $\therefore R$ is transitive.
- Hence R is an equivalence relation. The equivalence classes are
- $[0] = \{12k : k \in \mathbb{Z}\} \cup \{12k+6 : k \in \mathbb{Z}\}, = \{6k : k \in \mathbb{Z}\}$
- $[1] = \{12k+1 : k \in \mathbb{Z}\} \cup \{12k+5 : k \in \mathbb{Z}\}$,
- $[2] = \{12k+2 : k \in \mathbb{Z}\} \cup \{12k+4 : k \in \mathbb{Z}\}$, & $[3] = \{12k+3 : k \in \mathbb{Z}\}$.

- 5(a) F is a function if every element of F is an ordered pair and $\langle x, y \rangle \in F \wedge \langle x, z \rangle \in F \Rightarrow y = z$.
- (b) $f(x) = (3x+1)/(x-4) = 3 + 13/(x-4)$. Now suppose $f(x_1) = f(x_2)$.
 Then $3 + 13/(x_1-4) = 3 + 13/(x_2-4)$. $\therefore \frac{13}{x_1-4} = \frac{13}{x_2-4}$
 $\therefore x_1-4 = x_2-4$. So $x_1 = x_2$. Hence f is injective.
- Now let y be any element of $\mathbb{R} - \{3\}$. Choose $x = 4 + \frac{13}{y-3}$.
 Then $x \in \mathbb{R} - \{4\}$ and $f(x) = 3 + \frac{13}{x-4} = 3 + \frac{13}{\{13/(y-3)\}} = y$.
- Hence f is surjective. $\therefore f$ is a bijection.

- 6(a) $\bigcup_{i \in I} A_i = \{x : (\exists i \in I)(x \in A_i)\}$, $\bigcap_{i \in I} A_i = \{x : (\forall i \in I)(x \in A_i)\}$.
- (b) NO. Take $I = \{1, 2\}$, $A_1 = \{3, 4\}$, $A_2 = \{5, 6\}$, $B_1 = \{5, 7\}$
 and $B_2 = \{4, 8\}$. Then $A_1 \cap B_1 = \emptyset$ and $A_2 \cap B_2 = \emptyset$
 So $\bigcup_{i \in I} (A_i \cap B_i) = (A_1 \cap B_1) \cup (A_2 \cap B_2) = \emptyset \cup \emptyset = \emptyset$.
 Also $A_1 \cup A_2 = \{3, 4, 5, 6\}$ and $B_1 \cup B_2 = \{4, 5, 7, 8\}$.
 Thus $(\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i) = (A_1 \cup A_2) \cap (B_1 \cup B_2) = \{4, 5\}$. Hence
 it is not always true that $\bigcup_{i \in I} (A_i \cap B_i) = (\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i)$.