

Answer all 6 questions. An unjustified answer will receive little or no credit. BEGIN EACH QUESTION ON A SEPARATE PAGE.

- (20) 1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} 2x & \text{if } x \neq 1 \\ \frac{x-1}{2} & \text{if } x=1. \end{cases}$
- (a) Find $(f \circ f)(x)$.
- (b) Find $f^{-1}(x)$.

- (15) 2. (a) Let $f: X \rightarrow Y$ be a function; $A, B \subseteq X$; and $C, D \subseteq Y$.

Define what are $f[A]$ and $f^{-1}[C]$

- (b) Is it always true that $f^{-1}[C] \cap f^{-1}[D] \subseteq f^{-1}[C \cap D]$?
- (c) Is it always true that $f[A] \cap f[B] \subseteq f[A \cap B]$?

- (15) 3. Use Mathematical Induction to prove that for each $n \in \mathbb{Z}^+$,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

- (20) 4(a) Define what is a finite set & what is a countable set.

- (b) Prove that $\mathbb{Z}^+ \times \mathbb{Z}^+ \sim \mathbb{Z}^+$.

[If you claim that a function is bijective, you must verify this.]

- (15) 5(a) Define what it means for $\langle a_n \rangle_{n \in \mathbb{Z}^+}$ to be convergent

- (b) Suppose $\langle a_n \rangle_{n \in \mathbb{Z}^+}$ converges to A and c is a non-zero constant. Prove that $\langle c \cdot a_n \rangle_{n \in \mathbb{Z}^+}$ converges to $c \cdot A$.

- (15) 6(a) Define what it means for $\langle b_n \rangle_{n \geq 1}$ to be a Cauchy sequence.

- (b) Prove that if $\langle b_n \rangle_{n \geq 1}$ & $\langle c_n \rangle_{n \geq 1}$ are Cauchy sequences, then $\langle b_n + c_n \rangle_{n \geq 1}$ is also a Cauchy sequence.

1(a) If $x \neq 1$ or -1 , then $(f \circ f)(x) = f(f(x)) = f\left(\frac{2x}{x-1}\right) = 2 \cdot \left(\frac{2x}{x-1}\right) / \left(\frac{2x}{x-1} - 1\right)$
 $= \frac{4x}{x-1} \cdot \frac{x-1}{x+1} = \frac{4x}{x+1}$. $\therefore (f \circ f)(x) = \begin{cases} \frac{4x}{x+1} & \text{if } x \neq 1 \text{ or } -1 \\ 4 & \text{if } x = 1 \\ 2 & \text{if } x = -1. \end{cases}$
 If $x = 1$, then $f(f(x)) = f(f(1)) = f(2) = 4$
 If $x = -1$, then $f(f(x)) = f(f(-1)) = f(1) = 2$

(b) Let $y = f(x)$ and suppose $x \neq 1$. Then $y \neq 2$ and $f^{-1}(y) = x$. Since $f(x) = 2x/(x-1)$, $y = 2x/(x-1)$. $\therefore y(x-1) = 2x$ & so $x(y-2) = y$
 $\therefore x = y/(y-2)$. $\therefore f^{-1}(y) = y/(y-2)$ for $y \neq 2$. Also $f^{-1}(2) = 1$.
 $\therefore f^{-1}(y) = \begin{cases} y/(y-2) & \text{if } y \neq 2 \\ 1 & \text{if } y = 2. \end{cases}$ $\therefore f^{-1}(x) = \begin{cases} \frac{x}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$

2(a) $f[A] = \{f(x) : x \in A\}$ and $f^{-1}[C] = \{x \in X : f(x) \in C\}$.

(b) YES. Let $x \in f^{-1}[C] \cap f^{-1}[D]$. Then $x \in f^{-1}[C]$ and $x \in f^{-1}[D]$.
 So $f(x) \in C$ and $f(x) \in D$. $\therefore f(x) \in C \cap D$. $\therefore x \in f^{-1}[C \cap D]$
 Hence $f^{-1}[C] \cap f^{-1}[D] \subseteq f^{-1}[C \cap D]$.

(c) NO. Let $f: \{-1, 0, 1\} \rightarrow \mathbb{N}$ be defined by $f(k) = 4k^2 + 3$.
 Also let $A = \{-1, 0\}$ and $B = \{0, 1\}$. Then $A \cap B = \{0\}$ and
 $f[A] \cap f[B] = \{3, 7\} \not\subseteq \{3\} = f[A \cap B]$. So $f[A] \cap f[B]$ is not
 always a subset of $f[A \cap B]$.

3. Let $P(n)$ be the statement $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$.
 Then $P(1)$ is true because $1/\sqrt{1} = 1 \geq \sqrt{1}$. We will now
 show that $(\forall n \in \mathbb{Z}^+) [P(n) \rightarrow P(n+1)]$ is true. So fix n and
 suppose that $P(n)$ is true. Then $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$.
 So $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n} \cdot \sqrt{n+1} + 1}{\sqrt{n+1}}$
 $\geq \frac{\sqrt{n} \cdot \sqrt{n+1}}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$
 $\therefore P(n+1)$ is true.

So by the Principle of Math. Ind. for \mathbb{Z}^+ , we get $(\forall n \in \mathbb{Z}^+) P(n)$ is true.

4(a) The set A is finite if we can find a bijection $f: \{1, 2, 3, \dots, n\} \rightarrow A$ for some $n \in \mathbb{N}$. A is countable if A is finite or denumerable. A is denumerable if we can find a bijection $f: \mathbb{Z}^+ \rightarrow A$.

(b) Let $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ be defined by $f(\langle k, l \rangle) = 2^{k-1} \cdot (2l-1)$. Now suppose $f(\langle k_1, l_1 \rangle) = f(\langle k_2, l_2 \rangle)$. Then $2^{k_1-1} \cdot (2l_1-1) = 2^{k_2-1} \cdot (2l_2-1)$. So $k_1-1 = k_2-1 \Rightarrow k_1 = k_2$. $\therefore 2l_1-1 = 2l_2-1 \Rightarrow l_1 = l_2$. Hence $\langle k_1, l_1 \rangle = \langle k_2, l_2 \rangle$. So f is injective. Now let $n \in \mathbb{Z}^+$. Then by taking out all the powers of 2, n can be written in the form $n = 2^p \cdot q$, where q is odd. Let $k = p+1$ and $l = (q+1)/2$. Then $f(\langle k, l \rangle) = 2^{k-1} \cdot (2l-1) = 2^p \cdot q = n$. So f is surjective. Hence f is a bijection and so $\mathbb{Z}^+ \times \mathbb{Z}^+ \sim \mathbb{Z}^+$.

5(a) The infinite sequence $\langle a_n \rangle_{n \in \mathbb{Z}^+}$ is convergent if $(\exists L \in \mathbb{R}) (\forall \varepsilon > 0) (\exists N \in \mathbb{Z}^+) (\forall n \geq N) (|a_n - L| < \varepsilon)$.

(b) Let $\varepsilon > 0$ be given. Then $\varepsilon/|c| > 0$. Since $\langle a_n \rangle$ converges to A , we can find an $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|a_n - A| < \frac{\varepsilon}{|c|}$. So for all $n \geq N$, $|c \cdot a_n - c \cdot A| = |c \cdot (a_n - A)| = |c| \cdot |a_n - A| < |c| \cdot (\varepsilon/|c|) = \varepsilon$. $\therefore (\forall \varepsilon > 0) (\exists N \in \mathbb{Z}^+) (\forall n \geq N) (|c \cdot a_n - c \cdot A| < \varepsilon)$. $\therefore \langle c \cdot a_n \rangle$ converges to $c \cdot A$.

6(a) The infinite sequence $\langle a_n \rangle_{n \geq 1}$ is a Cauchy sequence if $(\forall \varepsilon > 0) (\exists N \in \mathbb{Z}^+) (\forall m, n \geq N) (|a_m - a_n| < \varepsilon)$.

(b) Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$. Since $\langle b_n \rangle$ and $\langle c_n \rangle$ are Cauchy sequences we can find $N_1 \in \mathbb{Z}^+$ and $N_2 \in \mathbb{Z}^+$ such that $(\forall m, n \geq N_1) (|b_m - b_n| < \varepsilon/2)$ and $(\forall m, n \geq N_2) (|c_m - c_n| < \varepsilon/2)$. Let $N = \max\{N_1, N_2\}$. Then for all $m, n \geq N$ we have

$$\begin{aligned} |(b_m + c_m) - (b_n + c_n)| &= |(b_m - b_n) + (c_m - c_n)| \\ &\leq |b_m - b_n| + |c_m - c_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$\therefore (\forall \varepsilon > 0) (\exists N \in \mathbb{Z}^+) (\forall m, n \geq N) (|(b_m + c_m) - (b_n + c_n)| < \varepsilon)$. Hence $\langle b_n + c_n \rangle_{n \geq 1}$ is a Cauchy sequence also.