

CHAPTER 0

RESF 306.05. c1

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①

1. (a)  $\{1, 2, 3, 4, 5\}$

(b)  $\{-5, -4, -3, -2, -1, 0, 1, 2\}$

(c)  $\{1, 2, 3, 4, 5\}$

(d)  $\{2, 3, 4\}$

2. (a)  $(1/2, 1)$

(b)  $[-1, 7]$

3. Let  $x \in A \cup (B \cap C)$ . Then  $x \in A$  or  $x \in B \cap C$ .

Now if  $x \in A$ , then  $x \in A \cup B$  and  $x \in A \cup C$ .

So  $x \in (A \cup B) \cap (A \cup C)$ . And if  $x \in B \cap C$ ,  
then  $x \in B$  and  $x \in C$ . So  $x \in A \cup B$  and  $x \in A \cup C$ .

Thus  $x \in (A \cup B) \cap (A \cup C)$ . Hence in either case

$x \in (A \cup B) \cap (A \cup C)$ . Thus  $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ .

Let  $x \in (A \cup B) \cap (A \cup C)$ . Then  $x \in A \cup B$  and  $x \in A \cup C$

Now either  $x \in A$  or  $x \notin A$ . If  $x \in A$ , then

$x \in A \cup (B \cap C)$ . And if  $x \notin A$ , then  $x \in B$   
(because  $x \in A \cup B$ ) and  $x \in C$  (because  $x \in A \cup C$ )

So  $x \in B$  and  $x \in C$ . Thus  $x \in B \cap C$  and

consequently  $x \in A \cup (B \cap C)$ . Hence in either

case  $x \in A \cup (B \cap C)$ . Thus  $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

It now follows by Theorem 0.1 that  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .

4. Let  $x \in A \setminus (B \cup C)$ . Then  $x \in A$  and  $x \notin B \cup C$ .

But if  $x \notin B \cup C$ , then  $x \notin B$  and  $x \notin C$ .

So  $x \in A$  and  $x \notin B$ . Also  $x \in A$  and  $x \notin C$ . Thus

$x \in A \setminus B$  and  $x \in A \setminus C$ . Hence  $x \in (A \setminus B) \cap (A \setminus C)$

Thus  $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$ .

4. Let  $x \in (A \setminus B) \cap (A \setminus C)$ . Then  $x \in A \setminus B$  and  $x \in A \setminus C$ . So  $x \in A$  and  $x \notin B$ , and  $x \in A$  and  $x \notin C$ . Thus  $x \in A$ . Also  $x \notin B$  and  $x \notin C$ . So  $x \notin B \cup C$ . Thus  $x \in A$  and  $x \notin B \cup C$ . Hence  $x \in A \setminus (B \cup C)$ . Thus  $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$ .

It now follows from Theorem 0.1 that  $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ .

5. Let  $x \in A \cap B$ . Then  $x \in A$  and  $x \in B$ . In particular  $x \in A$ . Thus  $A \cap B \subseteq A$ . Now let  $x \in A$ . Then " $x \in A$  or  $x \in C$ " is a true statement. So  $x \in A \cup C$ . Thus  $A \subseteq A \cup C$ . Hence  $A \cap B \subseteq A \subseteq A \cup B$ .

6. (a) Suppose  $A \subseteq B$ . Let  $x \in C \setminus B$ . Then  $x \in C$  and  $x \notin B$ . So  $x \notin A$  because  $A \subseteq B$ . Thus  $x \in C$  and  $x \notin A$ . Hence  $x \in C \setminus A$ . Thus  $C \setminus B \subseteq C \setminus A$ .  
 (b) If  $C \setminus B \subseteq C \setminus A$ , it does not follow that  $A \subseteq B$ . Counterexample: Take  $A = \{1, 2, 5\}$ ,  $B = \{1, 2, 3, 6\}$  and  $C = \{1, 2, 3, 4\}$

7.  $A \setminus (A \setminus B) = B$  if and only if  $B \subseteq A$ . To get this criterion, first show that  $A \setminus (A \setminus B) = A \cap B$ . So let's prove this.

Let  $x \in A \cap B$ . Then  $x \in A$  and  $x \in B$ . So  $x \notin (A \setminus B)$  because to get in  $A \setminus B$ , we need  $x \notin B$ . Thus  $x \in A$  and  $x \notin (A \setminus B)$ . So  $x \in A \setminus (A \setminus B)$ . Hence  $A \cap B \subseteq A \setminus (A \setminus B)$ . Now let  $x \in A \setminus (A \setminus B)$ .

7. Then  $x \in A$  and  $x \notin (A \setminus B)$ . Now if  $x \in (A \setminus B)$ , then  $x \in A$  and  $x \notin B$ . So if  $x \notin A \setminus B$ , then  $x \notin A$  or  $x \in B$ . But we already know that  $x \in A$ , so we must have  $x \in B$ . Thus  $x \in A$  and  $x \in B$ . So  $x \in A \cap B$ . Hence  $A \setminus (A \setminus B) \subseteq A \cap B$ . Thus  $A \setminus (A \setminus B) = A \cap B$ .

Now it is an easy matter to prove that  $A \cap B = B$  if and only if  $B \subseteq A$ .

8. Let  $x \in (A \setminus B) \cup (B \setminus A)$ . Then  $x \in A \setminus B$  or  $x \in B \setminus A$ . Now if  $x \in A \setminus B$ , then  $x \in A$  and  $x \notin B$ . So  $x \in A \cup B$  (because  $x \in A$ ). Also  $x \notin A \cap B$  (because  $x \notin B$ ). Thus  $x \in (A \cup B) \setminus (A \cap B)$ . And if  $x \in B \setminus A$ , then  $x \in B$  and  $x \notin A$ . So  $x \in A \cup B$  (because  $x \in B$ ). Also  $x \notin A \cap B$  (because  $x \notin A$ ). Thus  $x \in (A \cup B) \setminus (A \cap B)$ .

So in either case  $x \in (A \cup B) \setminus (A \cap B)$ . Thus  $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$ .

Let  $x \in (A \cup B) \setminus (A \cap B)$ . Then  $x \in A \cup B$  and  $x \notin A \cap B$ . Forget for a minute that  $x \notin A \cap B$  and concentrate on the fact that  $x \in A \cup B$ .

Now if  $x \in A$ , then  $x \notin B$  because  $x \notin A \cap B$ . So  $x \in A \setminus B$ . Thus  $x \in (A \setminus B) \cup (B \setminus A)$ .

And if  $x \in B$ , then  $x \notin A$  because  $x \notin A \cap B$ . So  $x \in B \setminus A$ . Thus  $x \in (A \setminus B) \cup (B \setminus A)$ .

So in either case  $x \in (A \setminus B) \cup (B \setminus A)$ . Thus  $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$ .  $\therefore (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ .

10. (a)  $\{0\}$   
 (c)  $[0, 1]$

- (b)  $\mathbb{R}$   
 (d)  $(-1, 3)$

(4)

11. Let  $x \in S - (\bigcap_{\lambda \in \Lambda} A_\lambda)$ . Then  $x \in S$  and  $x \notin \bigcap_{\lambda \in \Lambda} A_\lambda$ . So  $x \in S$  and  $x \notin A_{\lambda_0}$  for some  $\lambda_0 \in \Lambda$ . Thus  $x \in S - A_{\lambda_0}$ . So  $x \in \bigcup_{\lambda \in \Lambda} (S - A_\lambda)$ . Thus  $S - (\bigcap_{\lambda \in \Lambda} A_\lambda) \subseteq \bigcup_{\lambda \in \Lambda} (S - A_\lambda)$ .

Now do the other half to complete the proof.

12. (a)  $\mathbb{R} - \{0\}$

- (b)  $\mathbb{R} - [1, 2]$

13. (a)  $\text{im}(f) = \{x \in \mathbb{N} : x \text{ is odd}\}$

- (b)  $f$  is 1-1

- (c)  $f$  is not onto

- (d)  $\text{dom}(f) = \text{im}(f)$

- (e)  $f^{-1}(n) = (n+1)/2$

14. (a)  $\text{dom}(f) = \mathbb{R} - \{-z\}$

- (b)  $\text{im}(f) = \mathbb{R} - \{1\}$

- (d)  $f^{-1}(x) = zx/(1-x)$

- (c)  $f$  is injective

15. Let  $f(x) = 3$  for each  $x \in A$ . Then  $f$  is not 1-1.

16. Let  $f(x) = x+1$ . Then  $f$  has an inverse.  $f^{-1}(x) = x-1$

17. Let  $f(x) = x+z$  and  $g$  be defined by

$$g(2) = g(3) = a, \quad g(4) = b, \quad g(5) = c, \quad g(6) = d, \quad g(7) = e.$$

Then  $g$  is not 1-1, but  $g \circ f$  is 1-1.

19. Hint: Suppose  $1+2+3+\dots+n = n(n+1)/z$ .

$$\text{Then } 1+2+3+\dots+n+(n+1) = n(n+1)/z + (n+1)$$

$$= (n+1)\left(\frac{n}{z} + 1\right) = \frac{(n+1)(n+z)}{z}.$$

(5)

20. Hint: Suppose  $1+3+5+\dots+(2n-1) = n^2$ . Then  
 $1+3+5+\dots+(2n-1)+(2n+1) = n^2 + 2n+1 = (n+1)^2$ .

21. Let  $P(n)$  be the statement " $n^3+5n$  is divisible by 6".  
Then  $P(0)$  is true because  $0^3+5 \cdot 0$  is divisible by 6.

Now suppose that  $P(n)$  is true. Then  $n^3+5n$  is divisible by 6. So

$$\begin{aligned}(n+1)^3 + 5 \cdot (n+1) &= n^3 + 3n^2 + 3n + 1 + 5n + 5 \\ &= (n^3 + 5n) + 6 \cdot \frac{n(n+1)}{2} + 6.\end{aligned}$$

Since each of the three terms on the right hand side is divisible by 6, it follows that  $(n+1)^3 + 5 \cdot (n+1)$  is divisible by 6. (Note  $n(n+1)/2$  is always an integer because either  $n$  is even or  $n+1$  is even.) Thus  $P(n+1)$  is true. Since  $n$  was arbitrary, it follows that  $P(n) \Rightarrow P(n+1)$  is true for each  $n \in \mathbb{N}$ .

By the 1st Principle of Mathematical Induction it follows that  $P(n)$  is true for all  $n \in \mathbb{N}$ . Thus  $n^3+5n$  is divisible by 6 for all  $n \in \mathbb{N}$ .

22. Hint: Use the fact that  $2n+1 < 2^n$  for all  $n \geq 3$  (see Example 0.11). Suppose  $n^2 < 2^n$  and  $n \geq 5$ .

$$\begin{aligned}\text{Then } (n+1)^2 &= n^2 + 2n + 1 \\ &< 2^n + 2n + 1 \\ &< 2^n + 2^n = 2^{n+1}.\end{aligned}$$

24. Hint: Suppose  $f(1) < 2^1$  &  $f(2) < 2^2$  & ... &  $f(n) < 2^n$ .  
Then  $f(n+1) = f(n) + f(n-1) + f(n-2) < 2^n + 2^{n-1} + 2^{n-2}$   
 $< 2^n + 2^{n-1} + 2^{n-1} = 2^n + 2^n = 2^{n+1}$ .

(6)

25. Hint: Suppose  $f(n) < 2.4$  and  $n \geq 2$ . Then

$$f(n+1) = \sqrt{3+f(n)} < \sqrt{3+2.4} = \sqrt{5.4} < \sqrt{5.76} = 2.4.$$

26. Hint: Suppose  $f(k) = -5 \cdot 3^k + 5^{k-1} + 2^{k+3}$  for  $1 \leq k \leq n$ .

$$\text{Then } f(n+1) = 8 \cdot f(n) - 15 \cdot f(n-1) + 6 \cdot 2^{n+1}$$

$$= 8 \cdot [-5 \cdot 3^n + 5^{n-1} + 2^{n+3}] - 15 \cdot [-5 \cdot 3^{n-1} + 5^{n-2} + 2^{n+2}] + 6 \cdot 2^{n+1}$$

$$= [-40 + 25] \cdot 3^n + [8 - 3] \cdot 5^{n-1} + [16 - 15 + 3] \cdot 2^{n+2}$$

$$= -5 \cdot 3^{n+1} + 5^n + 2^{n+4}.$$

27. Hint: Suppose  $f(k) = 5 \cdot 2^k + 2 \cdot (-3)^k$  for  $0 \leq k \leq n$ .

$$\text{Then } f(n+1) = 6 \cdot f(n-1) - f(n)$$

$$= 6 \cdot [5 \cdot 2^{n-1} + 2 \cdot (-3)^{n-1}] - [5 \cdot 2^n + 2 \cdot (-3)^n]$$

$$= (30 - 10) \cdot 2^{n-1} + (12 + 6) \cdot (-3)^{n-1}$$

$$= 5 \cdot 2^{n+1} + 2 \cdot (-3)^{n+1}.$$

31. Let  $f(n) = n$ , if  $n$  is odd; and  $f(n) = 1-n$ , if  $n$  is even.

30. Suppose  $A$  is a countable set and  $B \subseteq A$ .

If  $A$  is finite, then  $B$  is also finite and so is countable.

If  $A$  is not finite, then it is countably infinite. So there is a 1-1 function  $f: A \rightarrow \mathbb{N}$  which is onto. Define  $g: B \rightarrow$  by  $g(t) = t$  for each  $t \in B$ . Then  $g$  is 1-1 and hence  $B$  is equi-pollent to  $\text{im}(g)$ . Now  $\text{im}(g)$  is a subset of  $\mathbb{N}$ , so by Theorem 0.14  $\text{im}(g)$  is countable. Hence  $B$  must also be countable.

31. Let  $f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ 1-n & \text{if } n \text{ is even} \end{cases}$

Then  $f$  is 1-1 and onto. Remember  $\mathbb{J}$  is the set of positive integers

32. The proof is by induction on  $n$ .  $P_0$  is the set of all constant polynomials with integer coefficients. So  $P_0$  is just  $\mathbb{Z}$  and hence is countable.

Now suppose  $P_n$  is countable. Let  $f: \mathbb{Z} \times P_n \rightarrow P_{n+1}$  be defined by

$$f(k, p(x)) = x \cdot p(x) + k$$

Then  $f$  is a 1-1 and onto function. So  $P_{n+1}$  has the same cardinality as  $\mathbb{Z} \times P_n$ . But by Theorem 0.16  $\mathbb{Z} \times P_n$  is countable. Hence  $P_{n+1}$  is countable.

Thus by the Principle of Math. Induction,  $P_n$  is countable for each  $n \in \mathbb{N}$ .

34. If  $S$  is countable, then it is equivalent to a subset of  $J$ . Let  $f: S \rightarrow J$  be a function that gives this equivalence. Let  $\Lambda = \text{im}(f)$  and define the indexed family  $\{B_n\}_{n \in \Lambda}$  as follows:

$$B_n = A_k \text{ where } k \text{ is the value such that } f(k) = n.$$

Then  $\bigcup_{n \in \Lambda} B_n = \bigcup_{k \in S} A_k$ . But  $\bigcup_{n \in \Lambda} B_n$  is countable by Theorem 0.17. Hence  $\bigcup_{k \in S} A_k$  is countable

33. The set of all polynomials with integer coefficients is just  $\bigcup_{n \in \mathbb{N}} P_n$ . This is a countable union of countable sets. So by problem 34, it follows that  $\bigcup_{n \in \mathbb{N}} P_n$  is countable.

35. For each  $p \in P_n$ ,  $B(p)$  is finite because a polynomial of degree  $n$  has at most  $n$  real roots.

Since  $P_n$  is countable,  $\bigcup_{p \in P_n} B(p)$  is a countable union of finite sets. Hence  $\bigcup_{p \in P_n} B(p)$  is countable.

32. The proof is by induction on  $n$ .  $P_0$  is the set of all constant polynomials with integer coefficients. So  $P_0$  is just  $\mathbb{Z}$  and hence is countable.

Now suppose  $P_n$  is countable. Let  $f: \mathbb{Z} \times P_n \rightarrow P_{n+1}$  be defined by

$$f(\langle k, p(x) \rangle) = x \cdot p(x) + k$$

Then  $f$  is a 1-1 and onto function. So  $P_{n+1}$  has the same cardinality as  $\mathbb{Z} \times P_n$ . But by Theorem 0.16  $\mathbb{Z} \times P_n$  is countable. Hence  $P_{n+1}$  is countable.

Thus by the Principle of Math. Induction,  $P_n$  is countable for each  $n \in \mathbb{N}$ .

34. If  $S$  is countable, then it is equivalent to a subset of  $J$ . Let  $f: S \rightarrow J$  be a function that gives this equivalence. Let  $\Lambda = \text{im}(f)$  and define the indexed family  $\{B_n\}_{n \in \Lambda}$  as follows:

$$B_n = A_k \text{ where } k \text{ is the value such that } f(k) = n.$$

Then  $\bigcup_{n \in \Lambda} B_n = \bigcup_{k \in S} A_k$ . But  $\bigcup_{n \in \Lambda} B_n$  is countable by Theorem 0.17. Hence  $\bigcup_{k \in S} A_k$  is countable.

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Since  $P_n$  is countable,  $\bigcup_{p \in P_n} B(p)$  is a countable union of finite sets. Hence  $\bigcup_{p \in P_n} B(p)$  is countable.

35. Let  $Q_n = \bigcup_{p \in P_n} B(p)$  where  $B(p)$  and  $P_n$  are as in problem 35. Then the set of algebraic numbers is simply  $\bigcup_{n \in J} Q_n$ . But this is just a countable union of countable sets. Hence  $\bigcup_{n \in J} Q_n$  is countable. Thus the set of algebraic numbers is countable.

38. Let  $f(x) = c + (x-a)(d-c)/(b-a)$ . Then  $f$  is a 1-1 and onto function from  $[a, b]$  to  $[c, d]$ .

Hint:

$$39. \quad x = \frac{x}{2} + \frac{x}{2} < \frac{x}{2} + \frac{y}{2} = \frac{x+y}{2} = \frac{x}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2} = y$$

Hint:

$$40. \quad 0 \leq (\sqrt{x} - \sqrt{y})^2 = x - 2\sqrt{xy} + y.$$

$$\text{So } 2\sqrt{xy} \leq x+y. \text{ Thus } \sqrt{xy} \leq (x+y)/2.$$

44. Let  $\varepsilon > 0$  be given, and suppose  $x = \sup A$ .

Then  $x - \varepsilon < x$ . So  $x - \varepsilon$  is not an upper bound for  $A$  (because  $x$  was the least upper bound for  $A$ ). Hence there must be an  $a \in A$  such that  $x - \varepsilon < a$ . Also, since  $x$  was an upper bound for  $A$ , we know  $a \leq x$ . Thus  $x - \varepsilon < a \leq x$ .

45. Let  $\varepsilon > 0$  be given, and suppose  $y = \inf A$ . Then  $y + \varepsilon > y$ . So  $y + \varepsilon$  is not a lower bound for  $A$  (because  $y$  was the greatest lower bound for  $A$ ). Hence there must be an  $a \in A$  such that  $a < y + \varepsilon$ . Also, since  $y$  was a lower bound for  $A$ , we know  $y \leq a$ . Thus  $y \leq a \leq y + \varepsilon$ .

(9)

The basic idea in Problems 39-42 is to prove the required results by appealing only to the 12 Axioms for the Real Numbers on p. 23-24.

39. Suppose  $x < y$ . Since  $y/2 > 0$ , it follows from Axiom II that  $x \cdot y/2 < y \cdot y/2$ . So  $x/2 < y/2$

$$\begin{aligned} x &= x/2 + x/2 && \text{by Ax. 3} \\ &< x/2 + y/2 && \text{by Ax. 8} \\ &= y/2 + x/2 && \text{by Ax. 2} \\ &< y/2 + y/2 && \text{by Ax. 8 again} \\ &= y && \text{by Ax. 3} \end{aligned}$$

Since  $x/2 + y/2 = (x+y)/2$  from Ax. 3, it follows that

$$x < \frac{x+y}{2} < y$$

40. We know from Theorem 0.19 that for any  $x$ ,  $x^2 \geq 0$ .

$$\begin{aligned} \text{So } 0 &\leq (\sqrt{x} - \sqrt{y})^2 = \dots \text{ multiply \& simplify} \dots \\ &= x - 2\sqrt{xy} + y \end{aligned}$$

$$\text{So } 2\sqrt{xy} \leq x+y$$

$$\text{and hence } \sqrt{xy} \leq (x+y)/2$$

41. (i) Supp.  $0 < a < b$ . Then

$$0 = 0 \cdot a < a \cdot a < a \cdot b < b \cdot b \quad \text{by Ax. 11 used repeatedly}$$

$$\text{So } 0 < a^2 < b^2.$$

(ii) Hint: We know  $0 < \sqrt{a}$  &  $0 < \sqrt{b}$  by def. of " $\sqrt{\cdot}$ ".

Suppose  $\sqrt{b} \leq \sqrt{a}$  & get a contradiction

42. First prove  $x/y < a/b \Leftrightarrow xb < ay$ . Now use the facts that  $x(y+b) < (x+a)y$  &  $(x+a)b < a(y+b)$ .

37. Suppose  $A$  is equivalent to  $P(A)$ . Then we can find a bijection  $f : A \rightarrow P(A)$ . Now let

$$C = \{x \in A : x \notin f(x)\}$$

Then  $C$  is a subset of  $A$ . So  $C \in P(A)$ . Since  $f$  is a bijection, we can find an element  $a \in A$  such that  $f(a) = C$ . Now either  $a \in C$  or  $a \notin C$ . But if  $a \in C$ , then  $a \in f(a)$  [because  $C = f(a)$ ] and so  $a \notin C$  by the definition of  $C$ . And if  $a \notin C$ , then  $a \notin f(a)$  [because  $C = f(a)$ ] and so  $a \in C$  by the definition of  $C$ . Hence we can't  $a \in C$  and we can't have  $a \notin C$  which is a contradiction.  $\therefore A$  is not equiv. to  $P(A)$

43.  $A = \{r \in \mathbb{Q} : r^2 < 2\}$ . Suppose  $A$  has a largest element. Call it  $L$ . Then  $L \in \mathbb{Q}$ ,  $L^2 < 2$ , and for any element  $a \in A$ ,  $a \leq L$  (because  $L$  is the largest element of  $A$ ). Since  $i \in \mathbb{Q}$  &  $i^2 < 2$ ,  $i \in A$ . Hence  $i \leq L$ . Now let

$$\delta = \frac{2-L^2}{2L+1}$$

Then  $s \in \mathbb{Q}$  because  $2-L^2 \in \mathbb{Q}$  &  $2L+1 \in \mathbb{Q}$ .  
and  $s > 0$  because  $2-L^2 > 0$  &  $2L+1 > 0$ .

Let  $a_0 = L + \delta$ . Then  $a \in \mathbb{Q}$  and  $a > L$

$$\begin{aligned}
 \text{Also } a_0^2 &= (L+\delta)^2 = L^2 + 2\delta L + \delta^2 \\
 &= L^2 + \delta(2L + \delta) \\
 &< L^2 + \frac{2-L^2}{2L+1} \cdot (2L+1) \quad \leftarrow \text{bec. } \delta < 1 \\
 &\quad \& \delta = \frac{2-L^2}{2L+1} \\
 &= L^2 + (2 - L^2) = 2
 \end{aligned}$$

$\therefore a_0^2 < 2$ . So  $a_0 \in A$ . But  $a_0 > L$  which contradicts the def. of  $L$ . Hence  $A$  has no largest element.

# CHAPTER 1

(11)

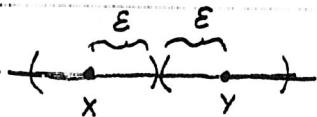
1. Take  $\varepsilon$  to be any positive number  $\leq 1/3$ .

2. Let  $\varepsilon = |x-y|/2$  and  $P = (x-\varepsilon, x+\varepsilon)$ ,  $Q = (y-\varepsilon, y+\varepsilon)$ .

Now if  $P \cap Q \neq \emptyset$ , then we can find a  $z \in P \cap Q$

$$\text{But then } |x-y| = |x-z+z-y|$$

$$\leq |x-z| + |y-z|$$



$$< \varepsilon + \varepsilon \quad \text{because } z \in P \text{ & } z \in Q.$$

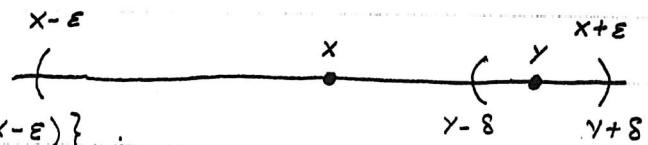
$$= 2\varepsilon = |x-y|$$

which is a contradiction. So  $P \cap Q = \emptyset$

3. Hint:

$$\text{Let } \delta = \min \{(x+\varepsilon)-y, y-(x-\varepsilon)\}.$$

$$\text{Then } (y-\delta, y+\delta) \subseteq (x-\varepsilon, x+\varepsilon).$$



4. For each  $n \in J$ ,  $\frac{3n+7}{n} = 3 + \frac{7}{n} > 3$ .

Also  $\frac{3n+7}{n} = 3 + \frac{7}{n} \leq 10$ . So a lower bound is 3 and an upper bound is 10.

5. Two examples are:  $\langle (-1)^n \rangle_{n=1}^{\infty}$  and  $\langle 2 + (-1)^n \rangle_{n=1}^{\infty}$ ,

6. (a) Let  $\varepsilon > 0$  be given. Take  $N = [1/\varepsilon] + 1$ . Then  $N > 1/\varepsilon$ , so  $1/N < \varepsilon$ .

So for all  $n \geq N$ ,  $|5 + 1/n - 5|$

$$= 1/n \leq 1/N < \varepsilon$$

Hence the sequence  $\langle 5 + 1/n \rangle_{n=1}^{\infty}$  converges to 5.

(b) Hint: Take  $N = [2/\varepsilon] + 1$ . Then  $N > 2/\varepsilon$ . So  $\frac{2}{N} < \varepsilon$ .

Thus  $\left| \frac{2-2n}{n} - (-2) \right| = \frac{2}{n} \leq \frac{2}{N} < \varepsilon$ , for all  $n \geq N$ .

6. (c) Hint: Take  $N = \lceil \frac{1}{\epsilon} \rceil + 1$ . Then  $N > \frac{1}{\epsilon}$ , so  $\frac{1}{N} < \epsilon$ .  
 Thus for all  $n \geq N$ ,  $\left| \frac{n+1}{n} - 1 \right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$ .

(d) Hint: Take  $N = \lceil \frac{3}{4\epsilon} \rceil + 1$ . Then  $N > \frac{3}{4\epsilon}$ , so  $\frac{3}{4N} < \epsilon$ .  
 Thus for all  $n \geq N$ ,  $\left| \frac{3n}{2n+1} - \frac{3}{2} \right| = \frac{3}{2(2n+1)} < \frac{3}{4n} \leq \frac{3}{4N} < \epsilon$ .

7. (a) Suppose  $\langle a_n \rangle$  converges to  $A$ . Let  $\epsilon > 0$  be given.  
 Then we can find an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - A| < \epsilon$ .

So for all  $n \geq N$ ,  $|(a_n - A) - 0| = |a_n - A| < \epsilon$ .

Thus  $\langle a_n - A \rangle$  converges to 0.

(b) The proof of the converse is similar.

8. Suppose  $\langle a_n \rangle$  converges to  $A$ . Let  $\epsilon > 0$  be given.  
 Then we can find an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - A| < \epsilon$ . So for all  $n \geq N$ ,

$$\begin{aligned} |b_n - A| &= \left| \frac{a_n + a_{n+1}}{2} - A \right| \\ &= \left| \frac{a_n - A}{2} + \frac{a_{n+1} - A}{2} \right| \\ &\leq \left| \frac{a_n - A}{2} \right| + \left| \frac{a_{n+1} - A}{2} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence  $\langle b_n \rangle$  converges to  $A$ .

9. Suppose  $\langle a_n \rangle$  converges to  $A$  and  $\langle b_n \rangle$  converges to  $A$ .  
 Let  $\epsilon > 0$  be given. Then we can find  $N_1 \in \mathbb{N}$ ,  $N_2 \in \mathbb{N}$   
 such that for all  $n \geq N_1$ ,  $|a_n - A| < \epsilon$   
 and for all  $n \geq N_2$ ,  $|b_n - A| < \epsilon$ . Let  $N = \max\{N_1, N_2\}$ .  
 Then for all  $n \geq N$ ,  $A - \epsilon < a_n < A + \epsilon$  and  $A - \epsilon < b_n < A + \epsilon$ .  
 So for all  $n \geq N$ ,  $A - \epsilon < a_n \leq c_n \leq b_n < A + \epsilon$ .  
 Thus for all  $n \geq N$ ,  $|c_n - A| < \epsilon$ . Hence  $\langle c_n \rangle$  converges to  $A$ .

(13)

10. (a) Hint: If  $|a_n - A| < \varepsilon$ , then  $|(a_n) - (A)| \leq |a_n - A| < \varepsilon$ .

(b) The converse is false. Look at  $\langle (-1)^n \rangle$ .

11. Let  $\varepsilon > 0$  be given. We know that there is an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a_n = \alpha$ .

So for all  $n \geq N$ ,  $|a_n - \alpha| = 0 < \varepsilon$ .

Hence  $\langle a_n \rangle$  converges to  $\alpha$ .

14. Hint: Take  $\varepsilon = 1$ . Then we can find an  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ ,  $|a_n - a_m| < \varepsilon = 1$ . So for all  $n \geq N$ ,  $|a_n - a_N| < 1$ . Let

$$M = \max \{a_1, a_2, a_3, \dots, a_{N-1}, a_N + 1\}$$

$$S = \min \{a_1, a_2, a_3, \dots, a_{N-1}, a_N - 1\}$$

Then for all  $n \in \mathbb{Z}$ ,  $S \leq a_n \leq M$ .

15. Let  $\varepsilon > 0$  be given. Since  $\langle a_n \rangle$  and  $\langle b_n \rangle$  are Cauchy, we can find  $N_1$  and  $N_2$  such that

$$|a_n - a_m| < \varepsilon/2 \quad \text{for all } m, n \geq N_1$$

$$\text{and } |b_n - b_m| < \varepsilon/2 \quad \text{for all } m, n \geq N_2$$

Let  $N = \max \{N_1, N_2\}$ . Then for all  $m, n \geq N$

$$\begin{aligned} |(a_n + b_n) - (a_m + b_m)| &= |(a_n - a_m) + (b_n - b_m)| \\ &\leq |a_n - a_m| + |b_n - b_m| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence  $\langle a_n + b_n \rangle$  is a Cauchy sequence.

16. Hint: First observe that Cauchy sequences are bounded.

So we can find  $M > 0$  such that for all  $n \geq 1$

$|a_n| \leq M$  and  $|b_n| \leq M$ . Now since  $\langle a_n \rangle$  and  $\langle b_n \rangle$  are Cauchy, we can find  $N_1$  and  $N_2$  such that

16

$$|a_n - a_m| < \varepsilon/2M \quad \text{for all } n, m \geq N,$$

$$\text{and } |b_n - b_m| < \varepsilon/2M \quad \text{for all } n, m \geq N_2.$$

Put  $N = \max\{N_1, N_2\}$ . Then

$$\begin{aligned} |a_n b_n - a_m b_m| &= |a_n b_n - a_n b_m + a_n b_m - a_m b_m| \\ &\leq |a_n| \cdot |b_n - b_m| + |a_n - a_m| \cdot |b_m| \\ &< M \cdot (\varepsilon/2M) + (\varepsilon/2M) \cdot M = \varepsilon. \end{aligned}$$

(14)

17 Hint:  $\left| \frac{2n+1}{n} - \frac{2m+1}{m} \right| = \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{m-n}{mn} \right| \leq \frac{m}{mn} < \frac{1}{n}$   
if we know that  $m \geq n$ .

18.  $\{1/n : n \geq 1\} \cup \{n/(n+1) : n \geq 1\}$  is one example.

19.  $\{k + 1/n : n \in J, k \in J\}$ . The set of accumulation points is  $J$ . (See problem 19 also.)

20.  $[0, 1]$  is one example. 21.  $\{2^n : n \in J\}$

22. (a) Suppose  $x \notin S$ . We have to show that  $x$  is an accumulation point of  $S$ . Let  $\varepsilon > 0$  be given.

Since  $x = \sup(S) = \text{least upper bound for } S$ ,  $x - \varepsilon$  is not an upper for  $S$ . So we can find  $a \in S$  such that  $x - \varepsilon < a < x$  (remember  $x \notin S$ )

Thus each neighbourhood of  $x$  contains a member of  $S$  which is different from  $x$ . Hence  $x$  must be an accumulation point of  $S$ .

(b) The case in which  $x = \inf(S)$  is similar. Just suppose  $x \notin S$  and so on.

25. Hint: Observe that  $b_n = (a_n + b_n) - a_n$  and that  $\langle a_n + b_n \rangle$  and  $\langle a_n \rangle$  are convergent sequences.

26 The sequences  $\langle -n \rangle_{n=1}^{\infty}$  and  $\langle n+y_n \rangle_{n=1}^{\infty}$  are unbounded, and hence not convergent. But  $\langle -n+n+y_n \rangle = \langle y_n \rangle$  which is convergent.

27 Observe that  $b_n = a_n b_n / a_n$  and that  $\langle a_n b_n \rangle$  is convergent and  $\langle a_n \rangle$  converges to  $A \neq 0$ .

28 Hint: Split into two cases: Case(i):  $a=0$ , Case(ii):  $a \neq 0$ .

Case(i): Let  $\epsilon > 0$  be given. Then there is an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - 0| < \epsilon^2$ . So for all  $n \geq N$ ,  $|\sqrt{a_n} - 0| = \sqrt{a_n} < \epsilon$ .

Case(ii): Let  $\epsilon > 0$  be given. Put  $\epsilon' = \epsilon\sqrt{a}$ . Then  $\epsilon' > 0$ . So there is an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - a| < \epsilon'$ . So for all  $n \geq N$

$$|\sqrt{a_n} - \sqrt{a}| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}} < \frac{\epsilon'}{\sqrt{a}} = \epsilon.$$

31 Hint: Let  $\epsilon > 0$  be given. Then we can find  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - A| < \epsilon/2$ . Also since  $\langle a_n - A \rangle$  converges to 0,  $a_n - A$  is bounded. So we can find an  $M > 0$  such that for all  $n \geq 1$ ,  $|a_n - A| \leq M$ .

Now let  $N_1 = \max\{N, [2MN/\epsilon] + 1\}$ . Then  $N_1 > 2MN/\epsilon$ . And for all  $n \geq N_1$ ,

$$\begin{aligned} |a_n - A| &= \left| \frac{a_1 + a_2 + \dots + a_N + \dots + a_n}{n} - A \right| \\ &\leq \underbrace{\left| \frac{a_1 - A}{n} \right| + \left| \frac{a_2 - A}{n} \right| + \dots + \left| \frac{a_N - A}{n} \right|}_{\leq M \cdot \frac{N}{n}} + \dots + \underbrace{\left| \frac{a_n - A}{n} \right|}_{\leq \frac{n-N}{n} \cdot \frac{\epsilon}{2}} \\ &\leq M \cdot \frac{N}{n} + \frac{n-N}{n} \cdot \frac{\epsilon}{2} \end{aligned}$$

$$\begin{aligned} \therefore |a_n - A| &\leq M \cdot \frac{N}{n} + \frac{n-N}{n} \cdot \frac{\epsilon}{2} < \frac{M \cdot N}{N_1} + \frac{\epsilon}{2} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

- 32 (a) 1      (b) 0      (c) 0  
 (d) 0      (e) -1/4      (f) 0.

33 By the formula in Example 0.10,  $a_n = 2 + (-1/2)^{n-1}$ .  
 So the sequences converges to 2.

34.  $\langle (-1)^{2n} \cdot (1 - 1/2^n) \rangle$  converges to 1  
 $\langle (-1)^{2n+1} \cdot (1 - 1/(2n+1)) \rangle$  converges to -1.

35 Let  $n_1 = \text{smallest integer such that } |a_{n_1} - x| < 1$ .  
 After defining  $n_1, \dots, n_k$  we define

$n_{k+1} = \text{smallest integer larger than } n_1, \dots, n_k$   
 such that  $|a_{n_{k+1}} - x| < 1/(k+1)$ .

The fact that  $x$  is an accumulation point of  $\{a_n : n \in J\}$  allows us to choose such integers.

The subsequence  $a_{n_1}, a_{n_2}, a_{n_3}, \dots$  converges to  $x$ .

36 Hint: Split into two cases.

Case (i):  $\{a_n : n \in J\}$  is finite. In this case infinitely many terms of the sequence must be the same. The subsequence consisting of these equal terms will be convergent.

Case (ii):  $\{a_n : n \in J\}$  is infinite. Since the sequence is bounded, the set  $\{a_n : n \in J\}$  is bounded, so by the Weierstrass thm this set has at least 1 limit point.

Now use problem #35

37 Hint: Let  $A = \inf \{a_n : n \in J\}$ . Show that  $\langle a_n \rangle$  converges to  $A$ . Problem 0.45 may be helpful

## Chapter 2

1. Take  $L = -4$ . Let  $\epsilon > 0$  be given. Choose  $\delta = \epsilon$ . Then  $\delta > 0$  and whenever  $0 < |x - (-2)| < \delta$  we have

$$\begin{aligned}|f(x) - L| &= |(x^2 - 4)/(x+2) - (-4)| \\ &= |(x-2) + 4| \\ &= |x+2| < \delta = \epsilon.\end{aligned}$$

Hence  $f(x)$  has a limit as  $x$  tends to  $-2$ . The limit is of course  $L = -4$ .

2. Hint: Take  $L = -5$ . Let  $\epsilon > 0$  be given. Choose  $\delta = \epsilon/2$

Then  $\delta > 0$  and whenever  $0 < |x - (-2)| < \delta$  we have

$$\begin{aligned}|f(x) - L| &= |(2x^2 + 3x - 2)/(x+2) - (-5)| = \dots = \\ &= 2|x+2| < 2\delta = \epsilon.\end{aligned}$$

3. One example is  $f(x) = \begin{cases} 1/(2x-1) & \text{if } x \neq 1/2 \\ 0 & \text{if } x = 1/2 \end{cases}$

4. One example is  $f(x) = \begin{cases} 1 & \text{if } x \geq -2 \\ 0 & \text{if } x < -2. \end{cases}$

5. Suppose  $L_1 \neq L_2$ . Take  $\epsilon = |L_1 - L_2|/2$ . Then  $\epsilon > 0$ .

Since  $L_1$  is a limit of  $f$  as  $x$  tends to  $x_0$ , we can find  $\delta_1 > 0$  such that

$$|f(x) - L_1| < \epsilon \quad \text{whenever } 0 < |x - x_0| < \delta_1 \text{ and } x \in D$$

Also since  $L_2$  is a limit of  $f$  as  $x$  tends to  $x_0$ , we can find  $\delta_2 > 0$  such that

$$|f(x) - L_2| < \epsilon \quad \text{whenever } 0 < |x - x_0| < \delta_2 \text{ and } x \in D.$$

Let  $\delta = \min \{\delta_1, \delta_2\}$ . Then  $\delta > 0$ . Since  $x_0$  was an accumulation point of  $D$  we can find a

5. point  $x_1 \in D$  such that  $0 < |x_1 - x_0| < \delta$ . Hence

$$\begin{aligned}|L_1 - L_2| &= |L_1 - f(x_1) + f(x_1) - L_2| \\&\leq |f(x_1) - L_1| + |f(x_1) - L_2| \\&< \varepsilon + \varepsilon \quad \text{bec. } 0 < |x_1 - x_0| < \delta \\&= 2\varepsilon = |L_1 - L_2|.\end{aligned}$$

Thus  $|L_1 - L_2| < |L_1 - L_2|$  — a contradiction. Hence we must have  $L_1 = L_2$ .

6. No. Consider the sequence  $\langle y_n \pi \rangle$ . Then

$\langle f(y_n \pi) \rangle = \langle \cos(n\pi) \rangle$  which does not converge.  
[ $\cos(n\pi)$  alternates between -1 and 1]. Hence  $f(x)$  cannot have a limit as  $x$  tends to 0.

7. Hint: Take  $L = 0$ . Let  $\varepsilon > 0$  be given. Choose  $\delta = \varepsilon$

Then  $\delta > 0$  and whenever  $0 < |x - 0| < \delta$  we have

$$\begin{aligned}|f(x) - L| &= \dots = |x| \cdot |\cos(\frac{1}{x})| \\&\leq |x| \quad \text{because } |\cos(\frac{1}{x})| \leq 1 \\&< \delta = \varepsilon.\end{aligned}$$

8. Hint: Take  $L = 2$ . Let  $\varepsilon > 0$  be given. Choose  $\delta = \varepsilon/2$ .

Then  $\delta > 0$  and whenever  $0 < |x - 1| < \delta$  we have

$$\begin{aligned}|f(x) - L| &= \dots = |x^2 - 1| \\&= |x-1| \cdot |x+1| \\&< 2|x-1| \quad \text{because } x \in (0, 1) \\&< 2\delta = \varepsilon.\end{aligned}$$

9. No. [Hint:  $f$  does not have a limit at  $x=1$  because the function "blows up" at 1. Look at Example 2.5 and mimic the idea used there.]

10. If  $f$  has a limit at  $0$  and  $\langle a_n \rangle$  is a sequence which converges to  $0$ , then  $\langle f(a_n) \rangle$  should converge to that limit. Now  $\langle 1/n \rangle$  converges to  $0$  and  $\langle f(1/n) \rangle = \langle (1/n)^m \rangle = \langle 1/n^m \rangle$  which converges to  $1$ . Hence the limit of  $f$  at  $0$  has to be  $1$ .

11. Hint: Use Theorem 2.1 and Exercise 9 of Chapter 1.

12. Hint: Use Theorem 2.1 and Exercise 10 of Chapter 1.

13. The function  $f(x) = x - [x]$  has a limit at  $x_0$  if and only if  $x_0$  is not an integer.

14.  $f$  will have a limit at  $x_0$  if and only if  $8x_0 = 2x_0^2 + 8$ . Roughly speaking, this is because any neighbourhood of  $x_0$  contains both rational and irrational points. A rigorous proof is of course required.

15. Hint: Look at the proof of the theorem in Ch. 1 which states that Cauchy sequences are convergent. The basic idea is as follows:

Let  $\langle a_n \rangle$  be any sequence in  $D - \{x_0\}$  converging to  $x_0$ . Let  $\epsilon > 0$  be given. Then we can find a nbhd  $Q$  of  $x_0$  such that

$$|f(x) - f(y)| < \epsilon \quad \text{whenever } x, y \in D \cap (Q - \{x_0\})$$

Also since  $\langle a_n \rangle \rightarrow x_0$ , there is a  $N \in \mathbb{N}$  such that  $a_n \in Q$  for all  $n \geq N$ .

Hence for all  $m, n \geq N$  we have  $a_n, a_m \in D \cap Q - \{x_0\}$ , and so  $|f(a_n) - f(a_m)| < \epsilon$ . Thus  $\langle f(a_n) \rangle$  is a Cauchy seq. and so is convergent. Now use Theorem 2.1.

16. Hint: The limit is  $-1/6$ . Use theorem 2.4

17. The function has a limit at  $x_0$  if and only if  $x_0$  is not an odd integer.

18. Hint: The limit is  $1/2$ .

Use the fact that  $\frac{\sqrt{1+x} - 1}{x} = \frac{\sqrt{1+x} - 1}{x} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$

$$= \frac{1}{\sqrt{1+x} + 1}.$$

19. Hint: The limit is  $-1/6$ . Use the fact that for  $x \neq 0$ ,  $f(x) = -1/(1/\sqrt{9-x} + 3)$ .

20. This is an easy problem.

use Theorem 2.1 and Theorem 1.12.

24. We shall prove the result in the case where  $f$  is increasing. (Make sure that you do the case where  $f$  is decreasing).

Let  $B = \sup\{f(x) : a \leq x < b\}$ . Let  $\epsilon > 0$  be given.

Then  $(B - \epsilon)$  is not an upper bound for  $\{f(x) : a \leq x < b\}$

So we can find  $x_1 \in [a, b)$  such that

$$(B - \epsilon) < f(x_1).$$

Let  $\delta = b - x_1$ . Then  $\delta > 0$  and whenever  $0 < |x - b| < \delta$  we have

$$\begin{aligned}|f(x) - B| &\leq |f(x_1) - B| && \text{bec. } f(x_1) \leq f(x) \leq B \\ &= B - f(x_1) < \epsilon.\end{aligned}$$

Hence  $f$  has a limit at  $b$ , namely  $B$ .

Also take  $A = \inf\{f(x) : a < x \leq b\}$ . Then we can show that the limit of  $f$  at  $a$  is  $A$ , similarly.

25. We have  $f: [a,b] \rightarrow \mathbb{R}$  and  $g$  is defined as follows:  

$$g(x) = \sup \{f(t) : a \leq t \leq x\}.$$

We are given that  $f$  has a limit at  $x_0$  and that  
 $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ . We want to prove that  $g$  has a  
limit at  $x_0$ .

Now by problem 24 we see that  $g$  will have a limit  
at  $a$  and at  $b$  because  $g$  is increasing. So we  
only have to prove the result for  $x_0$  with  $a < x_0 < b$ .

Let  $U(x_0) = \inf \{g(x) : a < x \leq b\}$  and

$$L(x_0) = \sup \{g(x) : a \leq x < x_0\}$$

We will show that  $L(x_0) = g(x_0) = U(x_0)$ . From  
this it readily follows that  $g$  has a limit at  $x_0$ .

Suppose  $L(x_0) < g(x_0)$ . Since  $f(x) \leq g(x)$ , by the def.  
of  $g$ , it follows that for all  $x$  with  $a \leq x < x_0$

$$f(x) \leq \sup \{g(x) : a \leq x < x_0\} = L(x_0)$$

So  $f(x_0) = \lim_{x \rightarrow x_0} f(x) \leq L(x_0)$ . But then

$$g(x_0) = \sup \{f(x) : a \leq x \leq x_0\} \leq L(x_0)$$

which contradicts the fact that  $L(x_0) < g(x_0)$ . Hence  
we must have  $L(x_0) = g(x_0)$ .

Similarly, if  $g(x_0) < U(x_0)$  we will get a contradiction.

Hence  $L(x_0) = g(x_0) = U(x_0)$  and we are done.

27. (a) The limit of  $f$  at  $x_0$  is  $+\infty$  iff for each  $M > 0$ ,  
there is a  $\delta > 0$  such that

$$f(x) > M \quad \text{whenever } 0 < |x - x_0| < \delta \text{ and } x \in [a, b]$$

(b) One example is  $f(x) = \begin{cases} 1/x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$ .  $[a, b] = [-1, 1]$

The limit of  $f$  at 0 is  $+\infty$ .

Ch.2 #26 We know that  $f(x+y) = f(x) \cdot f(y)$  for all  $x, y \in \mathbb{R}$  and that  $\lim_{x \rightarrow 0} f(x)$  exists. We want to prove that  $\lim_{x \rightarrow a} f(x)$  exists for all  $a \in \mathbb{R}$ . We also want to show that  $\lim_{x \rightarrow 0} f(x) = 0$ , or  $f(x) = 0$  for all  $x \in \mathbb{R}$ .

There are two cases: either  $f(0) = 0$  or  $f(0) \neq 1$ .

Suppose  $f(0) = 0$ . Then for each  $x \in \mathbb{R}$  we have

$$f(x) = f(x+0) = f(x) \cdot f(0) = 0$$

So  $\lim_{x \rightarrow a} f(x)$  exists for each  $a \in \mathbb{R}$ . Also  $f(x) = 0$  for all  $x \in \mathbb{R}$ .

So suppose  $f(0) \neq 0$ . Then  $f(0) = f(0+0) = f(0) \cdot f(0)$  and so  $f(0) = 1$ . Also for each  $x \in \mathbb{R}$

$$1 = f(0) = f(-x+x) = f(-x) \cdot f(x).$$

So  $f(x) \neq 0$  for each  $x \in \mathbb{R}$ . Moreover

$$f(1) = f(\frac{1}{2} + \frac{1}{2}) = f(\frac{1}{2}) \cdot f(\frac{1}{2}) = f(\frac{1}{2})^2 > 0,$$

and

$$f(1) = f\left(\frac{1}{2}\right)^2 = f\left(\frac{1}{4}\right)^4 = \dots = f\left(\frac{1}{2^n}\right)^{2^n}.$$

So  $f\left(\frac{1}{2^n}\right) = f(1)^{\frac{1}{2^n}}$ . Since  $\frac{1}{2^n} \rightarrow 0$  as  $n \rightarrow \infty$  it follows that

$$\lim_{x \rightarrow 0} f(x) = \lim_{n \rightarrow \infty} f\left(\frac{1}{2^n}\right) = \lim_{n \rightarrow \infty} f(1)^{\frac{1}{2^n}} = 1 \quad (\text{see Ch.1 #38})$$

Thus  $\lim_{x \rightarrow 0} f(x) = 1$ . Also  $\lim_{x \rightarrow a} f(x)$  exists for each  $a \in \mathbb{R}$  because

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} f(x-a+a) = \lim_{x \rightarrow a} f(x-a) \cdot f(a) \\ &= \lim_{y \rightarrow 0} f(y) \cdot f(a) = 1 \cdot f(a). \end{aligned}$$

So in the first case we got  $\lim_{x \rightarrow a} f(x)$  exist &  $f(x) = 0$  and in the second case we got  $\lim_{x \rightarrow a} f(x)$  exist &  $\lim_{x \rightarrow a} f(x) = 0$ .