

*Answer all 6 questions. No calculators, notes, or on-line stuff are allowed. An unjustified answer will receive little or no credit. Draw a line to separate each of your 6 solutions to the 6 questions.*

- (15) 1. (a) If  $R$  is a relation, define exactly when  $R$  is a *function*, and define exactly when  $R^{-1}$  is a *function*.  
(b) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function with  $f(x) = (3x+1)/(x-2)$  if  $x \neq 2$ ; &  $f(2) = 3$ . Find (with justification)  $f^{-1}(x)$  for each  $x \in \mathbb{R}$ .
- (20) 2. (a) Let  $g: X \rightarrow Y$  be a function and suppose that  $A, B \subseteq X$  and  $C, D \subseteq Y$ . Define what is  $g[A]$  and define what is  $g^{-1}[C]$ .  
(b) Is it always true that  $g[A \cap B] \subseteq g[A] \cap g[B]$ ?  
(c) Is it always true that  $g^{-1}[C - D] \subseteq g^{-1}[C] - g^{-1}[D]$ ?
- (15) 3. (a) Using quantifiers, write down the *First Principle of Math. Induction* for  $\mathbb{N}$ . Also define what is an *infinite sequence* of elements from a set  $S$ .  
(b) Prove that for each  $n \in \mathbb{N}$ ,  $4^n + 15n - 1$  is an *integer-multiple* of 9.
- (15) 4. (a) Define what it means for  $A$  to be *denumerable* and for  $B$  to be *finite*.  
(b) Prove that  $\mathbb{Z}^+ \times \mathbb{N}$  is a denumerable set. [If you claim that a function is a *bijection*, then you must prove that the function is indeed a bijection.]
- (20) 5. (a) By using quantifiers, define what is a *convergent sequence*  $\langle a_n \rangle_{n \in \mathbb{N}}$  of real numbers, and also what is a *bounded sequence*  $\langle c_n \rangle_{n \in \mathbb{N}}$  of real numbers.  
(b) Suppose  $\langle a_n \rangle_{n \in \mathbb{N}}$  converges to  $A$  and  $\langle b_n \rangle_{n \in \mathbb{N}}$  converges to  $B$ . Prove that  $\langle 3b_n + 2a_n \rangle_{n \in \mathbb{N}}$  converges to  $3B + 2A$ . (No theorems from class are allowed.)
- (15) 6. (a) Use quantifiers to define what is a *Cauchy sequence*  $\langle c_n \rangle_{n \in \mathbb{N}}$  of real numbers.  
(b) If  $\langle a_n \rangle_{n \in \mathbb{N}}$  is a convergent sequence, prove that  $\langle a_n \rangle_{n \in \mathbb{N}}$  is a *Cauchy sequence*.

$\in \forall \exists \Delta \oplus \subseteq \notin \subset \rightarrow \neg \neq \infty \emptyset \equiv \approx \leftrightarrow \times \aleph \vee \nabla \square \cong \perp \pm \geq \leq \circ \uparrow \downarrow \perp - \cup \cap \mathbb{R} \mathbb{Z} \langle \rangle \mathbb{N}$

1(a) The relation  $R$  is a function if  $(\forall a, b, c)[(a, b) \in R \wedge (a, c) \in R \rightarrow (b = c)]$ . The relation  $R^{-1}$  is a function if  $(\forall a, b, c)[(b, a) \in R \wedge (c, a) \in R \rightarrow (b = c)]$

(b)  $f(x) = \begin{cases} (3x+1)/(x-2) & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$ . Let  $y = f(x)$ . Then  $x = f^{-1}(y)$ . Now  $y = \frac{3x+1}{x-2}$   
 $\therefore x(y-3) = 3x+1$  and thus  $xy-3x = 2y+1$ .  $\therefore x(y-3) = 2y+1$   
and hence  $f^{-1}(y) = x = (2y+1)/(y-3)$ . Thus  $f^{-1}(x) = \frac{2x+1}{x-3}$  for  $x \neq 3$   
Also  $f^{-1}(3) = 2$  because  $f(2) = 3$ . Thus  $f^{-1}(x) = \begin{cases} (2x+1)/(x-3) & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$

2(a)  $g[A] = \{g(x) : x \in A\}$  and  $g^{-1}[C] = \{x \in X : g(x) \in C\}$

(b) YES. Let  $y \in g[A \cap B]$ . Then  $y = g(x)$  for some  $x \in A \cap B$ .  
So  $x \in A$  and  $x \in B$ . Hence  $y \in g[A]$  because  $x \in A$  and  
 $y \in g[B]$  because  $x \in B$ . Thus  $y \in g[A] \cap g[B]$ . Hence  
 $g[A \cap B] \subseteq g[A] \cap g[B]$  is always true.

(c) YES. Let  $x \in g^{-1}[C \cap D]$ . Then  $g(x) \in C \cap D$ . So  
 $g(x) \in C$  and  $g(x) \notin D$ . So  $x \in g^{-1}[C]$  and  $x \notin g^{-1}[D]$   
(because if  $x \in g^{-1}[D]$ , then we would get  $g(x) \in D$  and  
this would contradict  $g(x) \notin D$ ). Hence  $x \in g^{-1}[C] - g^{-1}[D]$ .  
Thus  $g^{-1}[C \cap D] \subseteq g^{-1}[C] - g^{-1}[D]$  is always true.

3(a) Let  $P(n)$  be a first-order formula with free variable  $n$ . Then  
 $\{P(0) \wedge (\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]\} \Rightarrow (\forall n \in \mathbb{N})[P_n]$

An infinite sequence of elements of  $S$  is just a function  $f: \mathbb{N} \rightarrow S$ .

(b) Let  $P(n)$  be the formula  $(\exists k \in \mathbb{N})[4^n + 15n - 1 = 9k]$ .

Since  $4^0 + 15(0) - 1 = 1 - 1 = 0 = 9(0)$ ,  $P(0)$  is true. Now suppose  $P(n)$  is true. Then  $4^n + 15n - 1 = 9k$  for some  $k \in \mathbb{N}$ . Hence

$$\begin{aligned} 4^{n+1} + 15(n+1) - 1 &= 4 \cdot (4^n + 15n - 1) - 4(15n - 1) + 15(n+1) - 1 \\ &= 4(9k) - 45n + 18 = 9(4k - 5n + 2). \quad \text{So } P(n) \rightarrow P(n+1). \end{aligned}$$

3(b) Hence  $P(0) \wedge (\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]$ . Thus  $(\forall n \in \mathbb{N})[P(n)]$ .  
 So for each  $n \in \mathbb{N}$ ,  $4^n + 15n - 1$  is an integer multiple of 9. (2)

4(a)  $A$  is denumerable if we can find a bijection  $f: A \rightarrow \mathbb{N}$ .

$A$  is finite if we can find a bijection  $g: B \rightarrow \mathbb{N}_k$  for some  $k \in \mathbb{N}$ .

(b) Let  $f: \mathbb{Z}^+ \times \mathbb{N} \rightarrow \mathbb{N}$  be defined by  $f(k, l) = 2^{k-1} \cdot (2l+1) - 1$ .

Now if  $f(k_1, l_1) = f(k_2, l_2)$ , then  $2^{k_1-1} \cdot (2l_1+1) - 1 = 2^{k_2-1} \cdot (2l_2+1) - 1$ .

So  $2^{k_1-1} = 2^{k_2-1} \Rightarrow k_1 = k_2$  [because  $(2l_1+1)$  &  $(2l_2+1)$  are odd]

Hence  $2^{k_1-1} \cdot (2l_1+1) = 2^{k_1-1} \cdot (2l_2+1) \Rightarrow 2l_1+1 = 2l_2+1 \Rightarrow l_1 = l_2$ . So

$f(k_1, l_1) = f(k_2, l_2) \Rightarrow (k_1, l_1) = (k_2, l_2)$ .  $\therefore f$  is an injection.

Also let  $n \in \mathbb{N}$ . Then  $n+1$  can be written in the form  $2^a \cdot (2b+1)$

with  $a, b \in \mathbb{N}$ . So  $n = 2^a \cdot (2b+1) - 1 = 2^{k-1} \cdot (2l+1) - 1$  where

$k = a+1$  and  $l = b$ . Hence  $f$  is surjective.  $\therefore \mathbb{Z}^+ \times \mathbb{N} \approx \mathbb{N}$

and hence  $\mathbb{Z}^+ \times \mathbb{N}$  is denumerable.

$\langle a_n \rangle$  is bounded if  $(\exists L \in \mathbb{R})(\exists U \in \mathbb{R})(\forall n \in \mathbb{N})[L \leq a_n \leq U]$ .

5(a)  $\langle a_n \rangle_{n \in \mathbb{N}}$  is convergent if  $(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|a_n - L| < \epsilon]$ .

(b) We want to show  $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|3b_n + 2a_n - 3B - 2A| < \epsilon]$

So fix  $\epsilon > 0$ . Then  $\epsilon/5 > 0$ . Since  $\langle a_n \rangle$  converges to  $A$  we can find  $N_1 \in \mathbb{N}$  such that  $(\forall n \geq N_1)[|a_n - A| < \epsilon/5]$ . Also since  $\langle b_n \rangle$  converges to  $B$ , we can find an  $N_2 \in \mathbb{N}$  such that  $(\forall n \geq N_2)[|b_n - B| < \epsilon/5]$ . Let  $N = \max\{N_1, N_2\}$ .

Then for  $(\forall n \geq N)$  we have

$$\begin{aligned} |(3b_n + 2a_n) - (3B + 2A)| &= |(3(b_n - B) + 3B) + (2(a_n - A) + 2A)| \\ &= |3(b_n - B) + 2(a_n - A)| \\ &\leq |3(b_n - B)| + |2(a_n - A)| \\ &= 3|b_n - B| + 2|a_n - A| \\ &< 3 \cdot \frac{\epsilon}{5} + 2 \cdot \frac{\epsilon}{5} = \frac{5\epsilon}{5} = \epsilon. \end{aligned}$$

$\therefore (\forall n \geq N)[|(3b_n + 2a_n) - (3B + 2A)| < \epsilon]$ . Since  $\epsilon$  was arb.

we get  $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|(3b_n + 2a_n) - (3B + 2A)| < \epsilon]$ .

Hence  $\langle 3b_n + 2a_n \rangle$  converges to  $3B + 2A$ .

(3)

6(a)  $\langle c_n \rangle_{n \in \mathbb{N}}$  is a Cauchy sequence if  
 $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{N} \text{ with } m, n \geq N)[|c_m - c_n| < \varepsilon]$ .

(b) Suppose  $\langle a_n \rangle$  is a convergent sequence. Then  
 $(\exists A \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|a_n - A| < \varepsilon]$ .

Now fix  $\varepsilon > 0$ . Then  $\varepsilon/2 > 0$ . So we can find an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - A| < \varepsilon/2$ . So for all  $m, n \geq N$ , we have

$$\begin{aligned}|a_m - a_n| &= |(a_m - A) + (A - a_n)| \\ &\leq |(a_m - A)| + |A - a_n| \\ &= |a_m - A| + |a_n - A| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

So  $(\exists N \in \mathbb{N})(\forall m, n \geq N)[|a_m - a_n| < \varepsilon]$ . Since  $\varepsilon$  was arb., it follows that  $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)[|a_m - a_n| < \varepsilon]$

Hence if  $\langle a_n \rangle$  is convergent, then  $\langle a_n \rangle$  is a Cauchy sequence.

END

### Additional Comments:

We could have said in #1(a) that  $R^{-1}$  is a function if

(\*)  $(\forall a, b, c)[(a, b) \in R^{-1} \wedge (a, c) \in R^{-1} \rightarrow (b = c)]$  but we have to define everything in terms of what we were given. And all we were given was that  $R$  is a relation.

We can convert (\*) into the correct answer by just replacing

$(a, b) \in R^{-1}$  by  $(b, a) \in R$  and  $(a, c) \in R^{-1}$  by  $(c, a) \in R$  to get

(\*\*)  $(\forall a, b, c)[(b, a) \in R \wedge (c, a) \in R \rightarrow (b = c)]$ . Of course,  $(\forall a, b, c)$  is an abbreviation of  $(\forall a)(\forall b)(\forall c)$ . END COMMENT.