MAA 3200 - INTROD TO ADVANCED MATH — FLORIDA INT'L UNIV.
TEST #1 - FALL 2024 — TIME: 75 min.

Answer *all 6* questions. ***No calculators, cell-phones, or notes are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

(15) 1(a) Translate the following argument into ***symbolic language***.
"Either Adam or Carl will win. If Adam wins, then Ben will not win. Therefore, if Ben wins, then Carl will win."
(b) Use a ***truth table*** to determine if this argument is ***logically valid*** and define what it means for a *propositional formula* to be a ***tautology***.

(15) 2(a) Define $(\forall x \in A)[P(x)]$ *and* $(\exists x \in B)[Q(x)]$ in terms of ***unbounded quantifiers***.
(b) Convert the formula $\neg(\exists y)(\forall z)[\{f(y) > g(z)\} \rightarrow \{(y + f(z) = 5) \wedge \neg(y < z)\}]$ into a *logically equivalent formula* in which no "$\neg$" sign ***governs*** a *quantifier* or a *connective*. [*Specify which logical law you use at each step.*]

(16) 3(a) Prove that $\neg(\forall x \in A)[P(x) \wedge Q(x)] \Leftrightarrow (\exists x \in A)[\{\neg P(x)\} \vee \{\neg Q(x)\}]$ by using the logical laws for ***unbounded quantifiers***.
(b) Let $\langle A_i : i \in I\rangle$ be an *indexed family* of sets. Define $\bigcap_{i\in I}(A_i)$ and $\bigcup_{i\in I}(A_i)$ and then use your definition to prove that $B - [\bigcap_{i\in I}(A_i)] = \bigcup_{i\in I}(B - A_i)$.

(16) 4(a) Let ***R & S*** be relations. Define $R \circ S$, S^{-1}, and define when ***S*** is a ***function***.
(b) Let R, S, and T be relations. Prove that $(R \circ S) \circ T = R \circ (S \circ T)$.
(c) Suppose G and H are functions. Prove that $G \circ H$ is also a function.

(18) 5(a) Define what is an ***equivalence relation*** R on a set A & what is $[a]_R$ for $a \in A$.
(b) Let R be the relation on $\mathbb{Z}$ defined by aRb *if* $(b^3 - a^3)$ is an *integer multiple* of 12. Prove that R is an ***equivalence relation*** on $\mathbb{Z}$ and specify **all** the ***equivalence classes*** into which R partitions $\mathbb{Z}$.

(20) 6(a) Let $f: A \rightarrow B$ be a ***total** function.* Define when exactly is f ***injective*** and when exactly is f ***surjective*** ?
(b) Let $f: \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{3\}$ be the total function defined by $f(x) = (3x-7)/(x-2)$. Prove that $f: \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{3\}$ is ***injective & surjective***, & then **find** $f^{-1}(x)$.

$\langle \in \forall \exists \Delta \oplus \subseteq \notin \subset \rightarrow \neg \neq \infty \varnothing \equiv \approx \leftrightarrow \times \aleph \surd \nabla \Leftrightarrow \Rightarrow \square \cong \perp \pm \geq \leq {}^\circ \uparrow\downarrow \perp - \cup \cap \mathbb{R} \mathbb{Z} \vee \wedge \rangle$ END

MAA 3200 - Introd to Adv. Math   Florida Int'l Univ.

Solutions to Test #1   Fall 2024

#1(a) Let A = Adam wins, B = Ben wins, and C = Carl win. The argument says $[(A \vee C) \wedge (A \to \neg B)] \Rightarrow (B \to C)$. or $[(A \vee C) \wedge (A \to \neg B)]\ \therefore\ (B \to C)$

(b)

A	B	C	$(A \vee C)$	$\wedge$	$(A \to (\neg B))$	$\longrightarrow$	$(B \to C)$
0	0	0	0	0	1	1	1
0	0	1	1	1	1	1	1
0	1	0	0	0	1	1	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	1	1
1	0	1	1	1	1	1	1
1	1	0	1	0	0	1	0
1	1	1	1	0	0	1	1

Since we got a tautology the argument is logically valid.

(c) A propositional formula is a tautology if its truth-values are always $\underline{1}$, no matter what may be the truth-values of its constituent statement letters.

#2 (a) $(\forall x \in A)[P(x)]$ means $(\forall x)[(x \in A) \to P(x)]$ & $(\exists x \in B)[Q(x)]$ means $(\exists x)[(x \in B) \wedge Q(x)]$

(b) $\neg(\exists y)(\forall z)[\{f(y) > g(z)\} \to \{(y + f(z) = 5) \wedge \neg(y < z)\}]$

$\Leftrightarrow (\forall y)\neg(\forall z)[\neg\{f(y) > g(z)\} \vee \{(y + f(z) = 5) \wedge \neg(y < z)\}]$

by the $\exists$-quantifier neg. law & conditional law

$\Leftrightarrow (\forall y)(\exists z)\neg[\neg\{f(y) > g(z)\} \vee \{(y + f(z) = 5) \wedge \neg(y < z)\}]$

by the $\forall$-quantifier negation law.

$\Leftrightarrow (\forall y)(\exists z)[\neg\neg\{f(y) > g(z)\} \wedge \neg\{(y + f(z) = 5) \wedge \neg(y < z)\}]$

by DeMorgan's law

$\Leftrightarrow (\forall y)(\exists z)[\{f(y) > g(z)\} \wedge \{\neg(y + f(z) = 5) \vee \neg\neg(y < z)\}]$

by Double negation law & De Morgan's law

$\Leftrightarrow (\forall y)(\exists z)[\{f(y) > g(z)\} \wedge \{\neg(y + f(z) = 5) \vee (y < z)\}]$

by Double negation law

$\Leftrightarrow (\forall y)(\exists z)[\{f(y) > g(z)\} \wedge \{(y + f(z) = 5) \to (y < z)\}]$

3(a) $\neg(\forall x \in A)[P(x) \wedge Q(x)] \Leftrightarrow \neg(\forall x)[(x \in A) \to \{P(x) \wedge Q(x)\}]$

$\Leftrightarrow (\exists x)\neg[\{\neg(x \in A)\} \vee \{P(x) \wedge Q(x)\}] \Leftrightarrow (\exists x)[\{\neg\neg(x \in A)\} \wedge \neg\{P(x) \wedge Q(x)\}]$

$\Leftrightarrow (\exists x)[(x \in A) \wedge \{\neg P(x) \vee \neg Q(x)\}] \Leftrightarrow (\exists x \in A)[\{\neg P(x)\} \vee \{\neg Q(x)\}]$.

#3(b) $\bigcap_{i\in I} A_i = \{x : (\forall i\in I)[x\in A_i]\}$, $\bigcup_{i\in I} A_i = \{x : (\exists i\in I)[x\in A_i]\}$

$x\in B-\left(\bigcap_{i\in I} A_i\right) \Leftrightarrow (x\in B)\wedge\{\neg(x\in \bigcap_{i\in I} A_i)\} \Leftrightarrow (x\in B)\wedge\{\neg(\forall i\in I)(x\in A_i)\}$
$\Leftrightarrow (x\in B)\wedge(\exists i\in I)[\neg(x\in A_i) \Leftrightarrow (\exists i\in I)[(x\in B)\wedge(x\notin A_i)]$
$\Leftrightarrow (\exists i\in I)[x\in(B-A_i)] \Leftrightarrow x\in\bigcup_{i\in I}(B-A_i)$.
Hence $B-\left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I}(B-A_i)$.

#4(a) $R\circ S = \{(a,c) : (\exists b)[(a,b)\in S \wedge (b,c)\in R\}$, $S^{-1} = \{(b,a) : (a,b)\in S\}$
S is a function if $(\forall a)(\forall b)(\forall c)\{[(a,b)\in S\wedge(a,c)\in S]\to(b=c)\}$
(b) $(c,d)\in(R\circ S)\circ T \Leftrightarrow (\exists b)[(a,b)\in T \wedge (b,d)\in(R\circ S)]$
$\Leftrightarrow (\exists b)[(a,b)\in T\wedge(\exists c)\{(b,c)\in S\wedge(c,d)\in R\}]$
$\Leftrightarrow (\exists c)[(\exists b)\{(a,b)\in T\wedge(b,c)\in S\}\wedge(c,d)\in R]$
$\Leftrightarrow (\exists c)[(a,c)\in S\circ T \wedge (c,d)\in R]$
$\Leftrightarrow (a,d)\in R\circ(S\circ T)$. $\therefore (R\circ S)\circ T = R\circ(S\circ T)$.

(c) Since G & H are relations, $G\circ H$ is automatically a relation.
Now suppose $(a,c_1)\in G\circ H$ and $(a,c_2)\in G\circ H$. Then
$(\exists b_1)[(a,b_1)\in H \wedge (b_1,c_1)\in G]$ & $(\exists b_2)[(a,b_2)\in H\wedge(b_2,c_2)\in G]$.
So $(a,b_1)\in H$ and $(a,b_2)\in H$. Since H is a function, $b_1=b_2$.
Hence $(b_1,c_1)\in G$ & $(b_1,c_2)\in G$. because $b_1=b_2$. Thus
$c_1=c_2$ because G is a function. So $[(a,c_1)\in G\circ H$
& $(a,c_2)\in(G\circ H)]\Rightarrow(c_1=c_2)$. Hence $G\circ H$ is a function.

#5(a) R is an equivalence on A if (i) $(\forall a\in A)[aRa]$,
(ii) $(\forall a,b\in A)[aRb\to bRa]$ & (iii) $(\forall a,b,c\in A)[\{aRb\wedge bRc\}\to aRc]$.
$[a]_R = \{x\in A : xRa\}$ is the equivalence class of a under R.
(b). Let $a\in\mathbb{Z}$. Then $a^3-a^3 = 0 = 12(0)$. So $(\forall a\in\mathbb{Z})[aRa]$
Now suppose aRb. Then $b^3-a^3 = 12(k)$ for some $k\in\mathbb{Z}$. So
$a^3-b^3 = -(b^3-a^3) = -12k = 12(-k)$. $\therefore (\forall a,b\in\mathbb{Z})[aRb\to bRa]$
Finally if $aRb\wedge bRc$, then $b^3-a^3 = 12k$ & $c^3-b^3 = 12l$ for
some $k,l\in\mathbb{Z}$. $\therefore c^3-a^3 = (c^3-b^3)+(b^3-a^3) = 12l+12k = 12(l+k)$.

#5(b) Hence $(\forall a,b,c \in \mathbb{Z})[[aRb \wedge bRc] \to aRc]$. So R is indeed an equivalence relation on $\mathbb{Z}$.

$0^3 \equiv_{12} 0$, $3^3 \equiv_{12} 27 \equiv_{12} 3$, $6^3 \equiv_{12} 36(6) \equiv_{12} 0$, $9^3 \equiv_{12} 81(9) \equiv_{12} 9$

$1^3 \equiv_{12} 1$, $4^3 \equiv_{12} 16(4) \equiv_{12} 4(4) \equiv_{12} 4$, $7^3 \equiv_{12} 49(7) \equiv_{12} 7$, $10^3 \equiv_{12} 100(10) \equiv_{12} 4(10) \equiv_{12} 4$

$2^3 \equiv_{12} 8$, $5^3 \equiv_{12} 25(5) \equiv_{12} 1(5) \equiv_{12} 5$, $8^3 \equiv_{12} 64(8) \equiv_{12} 32 \equiv_{12} 8$, $11^3 \equiv_{12} 121(11) \equiv_{12} 1(11) \equiv_{12} 11$.

So the equivalence classes are:

$[0]_R = [0]_{12} \cup [6]_{12} = \{12k : k\in\mathbb{Z}\} \cup \{12k+6 : k\in\mathbb{Z}\}$, $[1]_R = [1]_{12} = \{12k+1 : k\in\mathbb{Z}\}$

$[2]_R = [2]_{12} \cup [8]_{12} = \{12k+2 : k\in\mathbb{Z}\} \cup \{12k+8 : k\in\mathbb{Z}\}$, $[3]_R = [3]_{12} = \{12k+3 : k\in\mathbb{Z}\}$

$[4]_R = [4]_{12} \cup [10]_{12} = \{12k+4 : k\in\mathbb{Z}\} \cup \{12k+10 : k\in\mathbb{Z}\}$, $[5]_R = [5]_{12} = \{12k+5 : k\in\mathbb{Z}\}$

$[7]_R = [7]_{12} = \{12k+7 : k\in\mathbb{Z}\}$, $[9]_R = [9]_{12} = \{12k+9 : k\in\mathbb{Z}\}$, $[11]_R = [11]_{12} = \{12k+11 : k\in\mathbb{Z}\}$

#6 (a) $f: A \to B$ is <u>injective</u> if $(\forall a_1, a_2 \in A)[\{f(a_1) = f(a_2)\} \to (a_1 = a_2)]$

$f: A \to B$ is <u>surjective</u> if $(\forall b \in B)(\exists a \in A)[f(a) = b]$.

(b) (i) $f(x) = \frac{3x-7}{x-2} = \frac{3(x-2)-1}{x-2} = 3 - \frac{1}{x-2}$. Now suppose $f(x_1) = f(x_2)$

Then $3 - \frac{1}{x_1-2} = 3 - \frac{1}{x_2-2}$. So $\frac{-1}{x_1-2} = \frac{-1}{x_2-2} \Rightarrow x_1 - 2 = x_2 - 2 \Rightarrow x_1 = x_2$

Hence f is injective

(ii) Now let $y \in \mathbb{R} - \{3\}$. Then $y \neq 3$. We will find an $x \in \mathbb{R} - \{2\}$ such that $f(x) = y$. Now if $y = f(x) = 3 - \frac{1}{x-2}$, then $\frac{1}{x-2} = 3 - y$

So $x - 2 = \frac{1}{3-y}$ and thus $x = 2 + \frac{1}{3-y}$. Let us now check that $f(x) = y$. We have $f(x) = 3 - \frac{1}{x-2} = 3 - \frac{1}{\left(2 + \frac{1}{3-y}\right) - 2}$

$= 3 - \frac{1}{1/(3-y)} = 3 - (3-y) = y$.

Since $\frac{1}{3-y}$ is never 0, x is never 2, so $x \in \mathbb{R} - \{2\}$

Hence f is surjective.

(iii) Since f is both injective & surjective, f is a bijection. Hence f^{-1} is a function. Let $y = f^{-1}(x)$. Then

$f(y) = f(f^{-1}(x)) = x$. But $f(y) = 3 - \frac{1}{y-2}$, so $3 - \frac{1}{y-2} = x$

$\therefore$ $3 - x = \frac{1}{y-2}$ and so $\frac{1}{3-x} = y - 2$. $\therefore$ $y = 2 + \frac{1}{3-x}$

But $y = f^{-1}(x)$ $\therefore$ $f^{-1}(x) = \boxed{2 + \frac{1}{3-x}} = \frac{(6-2x)+1}{3-x} = \frac{7-2x}{3-x}$

$= \boxed{\frac{2x-7}{x-3}}$

<u>END</u>