MAA 3200 - INTROD TO ADVANCED MATH — FLORIDA INT'L UNIV.

TEST #2 - FALL 2024 — TIME: 75 min.

Answer all 6 questions. ***No calculators, notes, or cell-phones are allowed.*** *An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

(20) 1. (a) Let $f: X \to Y$ be a function and suppose that $A, B \subseteq X$ and $C, D \subseteq Y$. Define what is $f[A]$ *and* define what is $f^{-1}[C]$.

(b) Is it *always* true that $f[A] - f[B] \subseteq f[A - B]$?

(c) Is it *always* true that $f^{-1}[C \cap D] \subseteq f^{-1}[C] \cap f^{-1}[D]$?

(15) 2. (a) Using quantifiers, write down the *First Principle of Math Induction* for $\mathbb{N}$ & define what is an ***infinite sequence*** of elements from a non-empty set S.

(b) Prove that $(\forall n \in \mathbb{N})\,[\,3^{n+2} + 4^{2n+1}$ is an *integer-multiple* of 13].

(15) 3. (a) Define what it means for a set A to be ***finite*** & for a set B to be ***uncountable***.

(b) Prove that $\mathbb{Z}^{+} \times \mathbb{N}$ is *denumerable.* [*If you claim that a function f is a bijection, then you must prove that the function f is indeed a bijection.*]

(20) 4. (a) Using *quantifiers*, define what it means for the sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ of real numbers to be ***convergent***, and what it means for $\langle a_n \rangle_{n \in \mathbb{N}}$ to be ***bounded***.

(b) Suppose $\langle b_n \rangle_{n \in \mathbb{N}}$ converges to B, and $\langle c_n \rangle_{n \in \mathbb{N}}$ converges to C. Prove that $\langle b_n . c_n \rangle_{n \in \mathbb{N}}$ is convergent. (*You may use the fact that conv. seq. are bounded.*)

(15) 5. (a) Use *quantifiers* to define what is a ***Cauchy sequence*** $\langle c_n \rangle_{n \in \mathbb{N}}$ of real numbers.

(b) If $\langle a_n \rangle_{n \in \mathbb{N}}$ and $\langle b_n \rangle_{n \in \mathbb{N}}$ are *Cauchy sequences*, prove that $\langle 2a_n + 5b_n \rangle_{n \in \mathbb{N}}$ is also a *Cauchy sequence.* (*No theorems are allowed*)

(15) 6. (a) Let $f: \mathbb{R} \to \mathbb{R}$ be a *partial function* which is defined in a *punctured neighbourhood* of a. Define exactly what it means for $\lim_{x \to a} [f(x)]$ to ***exist***.

(b) Prove that $\lim_{x \to 4} [\,3 + x^2\,]$ exists. (*No theorems are allowed.*) **END**

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MAA 3200 – Introd. to Adv. Math Florida Int'l Univ.

Solutions to Test #2 Fall 2024

#1 (a) $f[A] = \{y \in Y : (\exists x \in A)[y = f(x)]\}$ & $f^{-1}[C] = \{x \in X : f(x) \in C\}$

(b) Yes. Let $y \in f[A] - f[B]$. Then $y \in f[A]$ & $y \notin f[B]$. So we can find an $x \in A$ such that $y = f(x)$. Now if $x \in B$, then we would have $f(x) = y \in f[B]$ – which would contradict $y \notin f[B]$. So $x \notin B$. Hence $x \in A - B$. $\therefore f(x) = y \in f[A-B]$. So $f[A] - f[B] \subseteq f[A-B]$

(c) Yes. Let $x \in f^{-1}[C \cap D]$. Then $f(x) \in C \cap D$. So $f(x) \in C$ & $f(x) \in D$. Thus $x \in f^{-1}[C]$ & $x \in f^{-1}[D]$. $\therefore x \in f^{-1}[C] \cap f^{-1}[D]$. So $f^{-1}[C \cap D] \subseteq f^{-1}[C] \cap f^{-1}[D]$.

#2 (a) Let $P(n)$ be any formula of First-order Logic with free variable n. Then $\{P(0) \wedge (\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]\} \Longrightarrow (\forall n \in \mathbb{N})[P(n)]$.

An infinite sequence of elements of S is just a function $f : \mathbb{N} \to S$.

(b) Let $P(n)$ be the formula $(\exists k \in \mathbb{Z})[3^{n+2} + 4^{2n+1} = 13k]$. Then $P(n)$ is a formula of F.O.L with free variable n. Also $P(0)$ is true because $3^{0+2} + 4^{2(0)+1} = 9 + 4 = 13 = 13(1)$.

Now suppose that $P(n)$ is true. Then $3^{n+2} + 4^{2n+1} = 13k$ for some $k \in \mathbb{Z}$. So $3^{(n+1)+2} + 4^{2(n+1)+1} = 3 \cdot 3^{n+2} + 16 \cdot 4^{2n+1}$ $= 3 \cdot [3^{n+2} + 4^{2n+1}] + 13 \cdot 4^{2n+1} = 3 \cdot [13k] + 13 \cdot 4^{2n+1} = 13[3k + 4^{2n+1}]$

So $P(n+1)$ is true. Since n was arbitrary, $(\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]$. By the First Principle of Mathematical Induction, it follows that $(\forall n \in \mathbb{N})[P(n)]$. Hence $(\forall n \in \mathbb{N})[3^{n+2} + 4^{2n+1}$ is an integer-mult. of 13$]$.

#3 (a) A is finite if there exists a bijection $f : A \to \mathbb{N}_k$ for some $k \in \mathbb{N}$. B is uncountable if B is not finite, and there is no bijection from B to $\mathbb{N}$ (which is saying that B is also not denumerable).

(b) Let $f : \mathbb{Z}^+ \times \mathbb{N} \to \mathbb{N}$ be defined by $f(k, \ell) = [2^{k-1} \cdot (2\ell + 1)] - 1$. We will show that f is injective & surjective. From this it will follow that f is a bijection – and thus $\mathbb{Z}^+ \times \mathbb{N}$ will be countable.

#3(b) Suppose $f(k_1,l_1)=f(k_2,l_2)$. Then $2^{k_1-1}\cdot(2l_1+1)-1=2^{k_2-1}\cdot(2l_2+1)-1$.
So $2^{k_1-1}\cdot(2l_1+1)=2^{k_2-1}\cdot(2l_2+1)$. Since $(2l_1+1)$ & $(2l_2+1)$ are both odd, it follows that $2^{k_1-1}=2^{k_2-1}$ & so $k_1-1=k_2-1 \Rightarrow k_1=k_2$.
Hence $2^{k_1-1}(2l_1+1)=2^{k_1-1}(2l_2+1)$. So $2l_1+1=2l_2+1 \Rightarrow l_1=l_2$.
Thus $f(k_1,l_1)=f(k_2,l_2) \Rightarrow (k_1,l_1)=(k_2,l_2)$. $\therefore$ f is injective.
Now suppose n is any element of $\mathbb{N}$. Then we can find $a,b\in\mathbb{N}$ such that $n+1=2^a\cdot(2b+1)$ by factoring-out all the powers of 2 in $(n+1)$. So $n=2^{a+1-1}\cdot(2b+1)-1=2^{k-1}\cdot(2l+1)-1$ if we put $k=a+1$ & $l=b$. $\therefore$ $n=f(k,l)$, so f is surjective.

#4(a) $\langle a_n\rangle_{n\in\mathbb{N}}$ is <u>conv.</u> if $(\exists L\in\mathbb{R})(\forall\varepsilon>0)(\exists N\in\mathbb{N})(\forall n\geq N)[|a_n-L|<\varepsilon]$.
$\langle a_n\rangle_{n\in\mathbb{N}}$ is <u>bounded</u> if $(\exists L\in\mathbb{R})(\exists U\in\mathbb{R})(\forall n\in\mathbb{N})[L\leq a_n\leq U]$.

(b) Suppose $\langle b_n\rangle_{n\in\mathbb{N}}$ converges to B. Then $\langle b_n\rangle_{n\in\mathbb{N}}$ is bounded.
So we can find an $M>0$ such that $(\forall n\in\mathbb{N})[|a_n|\leq M]$ (just take $M=\max\{|L|,|U|\}+1$ in the definition from part (a)).
Now fix any $\varepsilon>0$. Let $\varepsilon'=\varepsilon/(M+|C|)$. Then $\varepsilon'>0$.
Since $\langle b_n\rangle$ conv. to B, we can find $N_1\in\mathbb{N}$ such that $(\forall n\geq N_1)[|b_n-B|<\varepsilon']$. Also since $\langle c_n\rangle$ conv. to C, we can find $N_2\in\mathbb{N}$ such that $(\forall n\geq N_2)[|c_n-C|<\varepsilon']$. Let $N=\max\{N_1,N_2\}$.
Then $(\forall n\geq N)$ we have

$$\begin{aligned}|b_n\cdot c_n-B.C| &= |b_n\cdot c_n-b_n\cdot C+b_n\cdot C-B.C|\\ &= |b_n\cdot(c_n-C)+(b_n-B)\cdot C|\\ &\leq |b_n.(c_n-C)|+|(b_n-B)\cdot C|\\ &= |b_n|.|c_n-C|+|b_n-B|.|C|\\ &< M.\varepsilon'+|C|.\varepsilon'=(M+|C|).\varepsilon'=\varepsilon.\end{aligned}$$

So $(\forall n\geq N)[|b_n.c_n-B.C|<\varepsilon]$. Since $\varepsilon>0$ was arbitrary, it follows that $(\forall\varepsilon>0)(\exists N\in\mathbb{N})(\forall n\geq N)[|b_n.c_n-B.C|<\varepsilon]$
So it follows from the def. in part (a) that $\langle b_n\cdot c_n\rangle$ is convergent.

#5 (a) $\langle c_n \rangle$ is a Cauchy seq. if $(\forall \varepsilon>0)(\exists N\in\mathbb{N})(\forall m,n\geq N)[|a_m-a_n|<\varepsilon]$.

(b) Fix any $\varepsilon>0$. Then $\varepsilon/9>0$. Since $\langle a_n\rangle$ is a Cauchy seq., we can find an $N_1\in\mathbb{N}$ such that $(\forall m,n\geq N_1)[|a_m-a_n|<\varepsilon/9]$. Also since $\langle b_n\rangle$ is a Cauchy seq., we can find an $N_2\in\mathbb{N}$ such that $(\forall m,n\geq N_2)[|a_m-a_n|<\varepsilon/9]$. Let $N=\max\{N_1,N_2\}$. Then $(\forall m,n\geq N)$ we have

$$\begin{aligned}|(4a_m+5b_m)-(4a_n+5b_n)| &= |4(a_m-a_n)+5(b_m-b_n)|\\ \leq |4.(a_m-a_n)|+|5.(b_m-b_n)| &= 4.|a_m-a_n|+5.|b_m-b_n|\\ &< 4(\varepsilon/9)+5(\varepsilon/9)=\varepsilon.\end{aligned}$$

So $(\forall m,n\geq N)[|(4a_m+5b_m)-(4a_n+5b_n)|<\varepsilon]$. Since $\varepsilon>0$ was arbitrary it follows that
$(\forall\varepsilon>0)(\exists N\in\mathbb{N})(\forall m,n\geq N)[|(4a_m+5b_m)-(4a_n+5b_n)|<\varepsilon]$.
Hence $\langle 4a_n+5b_n\rangle$ is a Cauchy sequence.

#6 (a) $\lim_{x\to a} f(x)$ <u>exists</u> if
$(\exists L\in\mathbb{R})(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in\mathbb{R}$ with $0<|x-a|<\delta)[|f(x)-L|<\varepsilon]$.

(b) Let $L=19$. Fix any $\varepsilon>0$. Choose $\delta=\min\{1,\varepsilon/9\}$.
Then $(\forall x\in\mathbb{R}$ with $0<|x-4|<\delta)$ we have

$$\begin{aligned}|f(x)-L| &= |3+x^2-19| = |x^2-16|\\ &= |(x+4).(x-4)|\\ &= |x+4|.|x-4|\\ &< 9.|x-4| < 9.\delta \leq 9.\tfrac{\varepsilon}{9}=\varepsilon.\end{aligned}$$

$$\boxed{\begin{aligned}&|x-4|<\delta<1\\ \therefore\ &-1<x-4<1\\ \therefore\ &8-1<x+4<8+1\\ \therefore\ &|x+4|<9\end{aligned}}$$

Since $\varepsilon>0$ was arbitrary, it follows that
$(\exists L\in\mathbb{R})(\forall\varepsilon>0)(\exists\delta>0)(\forall x\in\mathbb{R}$ with $0<|x-4|<\delta)[|f(x)-L|<\varepsilon]$.
So $\lim_{x\to 4}(3+x^2)$ exists & $\lim_{x\to 4}(3+x^2)=L=19$.

END