

Answer *all* 6 questions. **No calculators, cell-phones, or notes are allowed.** An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (16) 1(a) Translate the following *argument* into **precise logical language**.
 "Either Amy or Carla will win. If Carla wins, then Beth will not win.
 Therefore, if Beth wins, then Amy will win."
 (b) Use a **truth table** to determine if this argument is **logically valid**.
 (c) Define what it means for a *propositional formula* to be **satisfiable**.
- (16) 2(a) Define $(\forall x \in A)[P(x)]$ and $(\exists x \in B)[Q(x)]$ in terms of **unbounded quantifiers**.
 (b) Convert the formula $\neg(\forall y)(\exists z)[\{g(z) < h(z)\} \rightarrow \{(y+z) < 4\} \wedge \neg(y=z)\}]$ into a **logically equivalent formula** in which no " $\rightarrow$ " sign **governs** a quantifier or a connective. [Specify which logical law you use at each step.]
- (16) 3(a) Prove that $\neg(\exists x \in A)[P(x) \wedge Q(x)] \Leftrightarrow (\forall x \in A)[\{\neg P(x)\} \vee \{\neg Q(x)\}]$ by using the logical laws for **unbounded quantifiers**.
 (b) Let $\langle A_i : i \in I \rangle$ be an *indexed family* of sets. Define $\bigcap_{i \in I} (A_i)$ and $\bigcup_{i \in I} (A_i)$ and then use your definition to prove that $D - [\bigcup_{i \in I} (A_i)] = \bigcap_{i \in I} (D - A_i)$.
- (16) 4(a) Let R & S be relations. Define $R \circ S$, R^{-1} , & define when R is a **function**.
 (b) Let S and T be *relations*. Prove that $(T \circ S)^{-1} = (S^{-1}) \circ (T^{-1})$.
 (c) Suppose F and G are *functions*. Prove that $G \circ F$ is also a function.
- (16) 5(a) Define what is an **equivalence relation** R on a set A & what is $[a]_R$ for $a \in A$.
 (b) Let R be the relation on $\mathbb{Z}$ defined by aRb if $(b^2 - a^2)$ is an **integer multiple** of **10**. Prove that R is an **equivalence relation** on $\mathbb{Z}$ and specify **all** the **equivalence classes** (using set-theoretical notation) into which R partitions $\mathbb{Z}$.
- (20) 6(a) Let $f: A \rightarrow B$ be a *partial function*. Define when exactly is f a **total function**, when exactly is f **injective**, and when exactly is f **surjective**?
 (b) Let $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{4\}$ be the *total function* defined by $f(x) = (4x-5)/(x-3)$. Prove that $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{4\}$ is **injective & surjective**, & then find $f^{-1}(x)$.
 END

#1(a) Let A = Amy wins, B = Beth wins & C = Carla wins. The argument says $[(A \vee C) \wedge (C \rightarrow B)] \Rightarrow (B \rightarrow A)$.

(b)

A	B	C	$(A \vee C)$	$(C \rightarrow B)$	$[(A \vee C) \wedge (C \rightarrow B)]$	$(B \rightarrow A)$
0	0	0	0	1	0	1
0	0	1	1	1	1	1
0	1	0	0	0	0	0
0	1	1	1	1	1	1
1	0	0	1	1	1	0
1	0	1	1	1	1	0
1	1	0	1	0	0	1
1	1	1	1	1	1	1

Since we got a tautology, the argument is logically valid.

(c) A propositional formula $\varphi(P_1, \dots, P_k)$ is satisfiable if there is at least one assignment of truth-values to $P_1, \dots, P_k$ that makes $\varphi(P_1, \dots, P_k)$ true.

#2(a) $(\forall x \in A)[P(x)]$ means $(\forall x)[(x \in A) \rightarrow P(x)]$ & $(\exists x \in A)[Q(x)]$ means $(\exists x)[(x \in A) \wedge Q(x)]$.

$$(b) \neg(\forall y)(\exists z)[\{g(z) < h(z)\} \rightarrow \{(y+z) < 4 \wedge \neg(y=z)\}]$$

$$\Leftrightarrow (\exists y)[\neg(\exists z)[\{g(z) < h(z)\} \rightarrow \{(y+z) < 4 \wedge \neg(y=z)\}]] \text{ by neg. } \forall\text{-law}$$

$$\Leftrightarrow (\exists y)(\forall z)\neg[\{g(z) < h(z)\} \rightarrow \{(y+z) < 4 \wedge \neg(y=z)\}] \text{ by neg } \exists\text{-law}$$

$$\Leftrightarrow (\exists y)(\forall z)[\neg[\neg\{g(z) < h(z)\} \vee \{(y+z) < 4 \wedge \neg(y=z)\}]] \text{ by conditional law}$$

$$\Leftrightarrow (\exists y)(\forall z)[\neg\neg\{g(z) < h(z)\} \wedge \neg\{(y+z) < 4 \wedge \neg(y=z)\}] \text{ by De Morgan's law}$$

$$\Leftrightarrow (\exists y)(\forall z)[\{g(z) < h(z)\} \wedge \{(y+z) < 4 \vee \neg\neg(y=z)\}] \text{ by double-neg. \& De Morgan's law}$$

$$\Leftrightarrow (\exists y)(\forall z)[\{g(z) < h(z)\} \wedge \{(y+z) < 4 \rightarrow (y=z)\}] \text{ by cond. \& double-neg law}$$

$$\#3(a) \neg(\exists x \in A)[P(x) \wedge Q(x)] \Leftrightarrow \neg(\exists x)[(x \in A) \wedge \{P(x) \wedge Q(x)\}] \text{ by def. of } (\exists x \in A)$$

$$\Leftrightarrow (\forall x)\neg[(x \in A) \wedge \{P(x) \wedge Q(x)\}] \text{ by neg. } \forall\text{-law}$$

$$\Leftrightarrow (\forall x)[\neg(x \in A) \vee \neg\{P(x) \wedge Q(x)\}] \text{ by De Morgan's law}$$

$$\Leftrightarrow (\forall x)[(x \in A) \rightarrow \neg\{P(x) \wedge Q(x)\}] \text{ by Conditional law}$$

$$\Leftrightarrow (\forall x \in A)[\neg\{P(x) \wedge Q(x)\}] \text{ by def. of } (\forall x \in A)$$

$$\Leftrightarrow (\forall x \in A)[\{\neg P(x)\} \vee \{\neg Q(x)\}] \text{ by De Morgan's law.}$$

$$(b) \bigcap_{i \in I} (A_i) = \{x : (\forall i \in I)[x \in A_i]\} \text{ \& } \bigcup_{i \in I} (A_i) = \{x : (\exists i \in I)[x \in A_i]\}.$$

$$\begin{aligned}
 x \in D - \bigcup_{i \in I} (A_i) &\Leftrightarrow (x \in D) \wedge \neg [x \in \bigcup_{i \in I} A_i] \Leftrightarrow (x \in D) \wedge \neg [\exists i \in I] [x \in A_i] \\
 &\Leftrightarrow (x \in D) \wedge (\forall i \in I) [\neg (x \in A_i)] \Leftrightarrow (\forall i \in I) [(x \in D) \wedge (x \notin A_i)] \\
 &\Leftrightarrow (\forall i \in I) [x \in (D - A_i)] \Leftrightarrow x \in \bigcap_{i \in I} (D - A_i). \therefore D - \bigcup_{i \in I} (A_i) = \bigcap_{i \in I} (D - A_i).
 \end{aligned}$$

#4(a) $R \circ S = \{(a, c) : (\exists b) [a, b \in S \wedge (b, c) \in R]\}$, $R^{-1} = \{(b, a) : (a, b) \in R\}$,
 S is a function if $(\forall a)(\forall b)(\forall c) [\{a, b \in S \wedge (a, c) \in S\} \rightarrow (b = c)]$.

$$\begin{aligned}
 (b) (c, a) \in (T \circ S)^{-1} &\Leftrightarrow (a, c) \in (T \circ S) \\
 &\Leftrightarrow (\exists b) [a, b \in S \wedge (b, c) \in T] \Leftrightarrow (\exists b) [(c, b) \in T^{-1} \wedge (b, a) \in S^{-1}] \\
 &\Leftrightarrow (c, a) \in (S^{-1}) \circ (T^{-1}). \text{ Hence } (T \circ S)^{-1} = (S^{-1}) \circ (T^{-1}).
 \end{aligned}$$

(c) Since G & F are relations, $G \circ F$ is automatically a relation.

Now suppose $(a, c_1) \in G \circ F$ and $(a, c_2) \in G \circ F$. Then

$$(\exists b_1) [a, b_1 \in F \wedge (b_1, c_1) \in G] \text{ \& \& } (\exists b_2) [a, b_2 \in F \wedge (b_2, c_2) \in G]$$

Since $(a, b_1) \in F$, $(a, b_2) \in F$, and F is a function, $b_1 = b_2$.

So now $(b_1, c_1) \in G$ & $(b_1, c_2) \in G$. Since G is a function, $c_1 = c_2$.

$\therefore [(a, c_1) \in G \circ F \text{ \& \& } (a, c_2) \in G \circ F] \Rightarrow (c_1 = c_2)$. Hence $G \circ F$ is a function.

#5(a) R is an equivalence relation on A if $(\forall a \in A) [aRa]$,
 $(\forall a, b \in A) [aRb \rightarrow bRa]$, and $(\forall a, b, c \in A) [(aRb \wedge bRc) \rightarrow aRc]$.
 $[a]_R = \{x \in A : xRa\}$ is the equivalence class of a under R .

(b) Let $a \in \mathbb{Z}$. Then $a^2 - a^2 = 0 = 10(0)$. So $(\forall a \in \mathbb{Z}) [aRa]$.

Now suppose aRb . Then $b^2 - a^2 = 10(k)$ for some $k \in \mathbb{Z}$. So

$$a^2 - b^2 = -(b^2 - a^2) = -10k = 10(-k) \text{ with } -k \in \mathbb{Z}. \text{ So } (\forall a, b \in \mathbb{Z}) [aRb \rightarrow bRa].$$

Finally, if aRb & bRc , then $b^2 - a^2 = 10k$ & $c^2 - b^2 = 10l$ for some

$$k, l \in \mathbb{Z}. \therefore c^2 - a^2 = (c^2 - b^2) + (b^2 - a^2) = 10l + 10k = 10(l+k) \text{ and}$$

so aRc . Thus $(\forall a, b, c \in \mathbb{Z}) [(aRb \wedge bRc) \rightarrow aRc]$. So R is an equiv. rel.

$$\begin{aligned}
 0^2 &\equiv_{10} 0, & 1^2 &\equiv_{10} 1, & 2^2 &\equiv_{10} 4, & 3^2 &\equiv_{10} 9, & 4^2 &= 16 \equiv_{10} 6, \\
 5^2 &= 25 \equiv_{10} 5, & 6^2 &= 36 \equiv_{10} 6, & 7^2 &= 49 \equiv_{10} 9, & 8^2 &= 64 \equiv_{10} 4, & 9^2 &= 81 \equiv_{10} 1.
 \end{aligned}$$

So the equivalence classes into which R partitions $\mathbb{Z}$ are:

$$[0]_R = [0]_{10} = \{10k : k \in \mathbb{Z}\}, [5]_R = [5]_{10} = \{10k+5 : k \in \mathbb{Z}\}, [1]_R = [9]_R = \{10k+1 : k \in \mathbb{Z}\}$$

$$[2]_R = [8]_R = \{10k+2 : k \in \mathbb{Z}\}, [3]_R = [7]_R = \{10k+3 : k \in \mathbb{Z}\}, \& [4]_R = [6]_R = \{10k+4 : k \in \mathbb{Z}\}.$$

#6(a) $f: A \rightarrow B$ is a total function if $(\forall a \in A)(\exists b \in B)[(a, b) \in f]$.

$f: A \rightarrow B$ is injective if $(\forall a_1, a_2 \in A)[\{f(a_1) = f(a_2)\} \rightarrow (a_1 = a_2)]$.

$f: A \rightarrow B$ is surjective if $(\forall b \in B)(\exists a \in A)[f(a) = b]$.

(b)(i) $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{4\}$ & $f(x) = (4x-5)/(x-3) = [4(x-3) + 7]/(x-3) = 4 + \frac{7}{x-3}$.

Now suppose $f(x_1) = f(x_2)$. Then $4 + 7/(x_1-3) = 4 + 7/(x_2-3)$.

So $7/(x_1-3) = 7/(x_2-3)$ & thus $(x_1-3)/7 = (x_2-3)/7$ & $x_1-3 = x_2-3$

Hence $x_1 = x_2$. So f is injective.

(ii) Now let $y \in \mathbb{R} - \{4\}$. We will find an $x \in \mathbb{R} - \{3\}$ such that $f(x) = y$.

Now if $y = f(x) = 4 + 7/(x-3)$, then $7/(x-3) = y-4$ & so $x-3 = 7/(y-4)$.

Thus x should be $3 + \frac{7}{y-4}$. This x is well-defined because $y \neq 4$

& it is in $\mathbb{R} - \{3\}$ because $7/(y-4) \neq 0$. Also we have

$$f(x) = 4 + \frac{7}{\left(3 + \frac{7}{y-4}\right) - 3} = 4 + 7/\left(\frac{7}{y-4}\right) = 4 + 7 \cdot \frac{y-4}{7} = 4 + (y-4) = y.$$

Hence $f(x) = y$ and thus f is surjective.

(iii) Since f is injective & surjective, f is a bijection. Hence

$f^{-1}: \mathbb{R} - \{4\} \rightarrow \mathbb{R} - \{3\}$ is a function. Let $x = f^{-1}(y)$. Now from

part (ii), we know $x = 3 + \frac{7}{y-4}$, so $f^{-1}(y) = 3 + \frac{7}{y-4}$.

$$\text{Thus } f^{-1}(x) = 3 + \frac{7}{x-4} = \frac{3(x-4) + 7}{x-4} = \boxed{\frac{3x-5}{x-4}}.$$

[Notice that f^{-1} is indeed defined for all $x \in \mathbb{R} - \{4\}$.]