

Answer all 6 questions. No calculators or Cell phones are allowed. An unjustified answer will receive little credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (15) 1(a) Translate the following argument into *symbolic language*. "Either Adam or Baalam will get married. If Adam gets married, then Caleb will not get married. Therefore if Caleb got married, then Baalam got married."
- (b) Use a *truth table* to determine if this argument is *logically valid*.
- (15) 2(a) Define what is $(\exists!x)P(x)$ and what is $(\forall x \in B)Q(x)$ in terms of *unbounded quantifiers*.
- (b) Convert $\neg (\forall x)(\exists y) [\{f(x,y) = 1\} \rightarrow \{(x = 2) \wedge (y \neq 0)\}]$ into a *logically equivalent* formula in which no " \neg " governs a *quantifier* or a *connective*.
- (15) 3(a) Define what is an *ordered pair* (a, b) in terms of sets.
- (b) Prove that for all sets A, B, & C we have $(A - B) \times C = (A \times C) - (B \times C)$.
- (20) 4(a) Let R & S be *relations*. Define what is the *composition* $R \circ S$.
- (b) Let R be the relation on the set of integers, \mathbb{Z} , defined by, aRb if $(a^3 - b^3)$ is an *integer multiple* of 8. Prove that R is an *equivalence relation* and find the *equivalence classes* into which R partitions \mathbb{Z} .
- (20) 5(a) Define what it means for the function $f: \mathbb{N} \rightarrow \mathbb{Q}$ to be *injective* and what it means for it to be *surjective*.
- (b) Let $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{5\}$ be the function with $f(x) = 5x / (x - 3)$. Prove that f is a *bijection*.
- (15) 6(a) Let $\langle A_i : i \in I \rangle$ be an *indexed family* of sets. Define what are $\bigcap_{i \in I} (A_i)$ and $\bigcup_{i \in I} (A_i)$.
- (b) Prove that for any set B, we have $B - \bigcup_{i \in I} (A_i) = \bigcap_{i \in I} (B - A_i)$.

1(a) Let $A = \text{Adam gets married}$, $B = \text{Bavalam gets married}$, & $C = \text{Caleb gets married}$. Argument says: $[(A \vee B) \wedge (A \rightarrow \neg C)] \rightarrow (C \rightarrow B)$

(b)	A	B	C	$[(A \vee B) \wedge (A \rightarrow \neg C)]$	$\rightarrow (C \rightarrow B)$
	T	T	T	T	T
	T	T	F	F	T
	T	F	T	F	F
	T	F	F	T	T
	F	T	T	T	T
	E	T	F	T	T
	F	E	T	F	F
	F	F	F	F	T

Since we got a tautology, the argument is logically valid

2(a) $(\exists !x) P(x)$ means $(\exists x) [P(x) \wedge (\forall y) \{P(y) \rightarrow y=x\}]$

$(\forall x \in B) Q(x)$ means $(\forall x) [x \in B \rightarrow Q(x)]$

$$\begin{aligned}
 (b) \quad & \neg (\forall x)(\exists y) [f(x,y)=1] \rightarrow \{ (x=2) \wedge (y \neq 0) \} && \text{quantifier neg. law} \\
 \Leftrightarrow & (\exists x) \neg (\exists y) [f(x,y)=1] \vee \{ (x=2) \wedge (y \neq 0) \} && [\neg (\exists y) \Leftrightarrow (\neg f(x,y)=1)] \\
 \Leftrightarrow & (\exists x) (\forall y) \neg [f(x,y)=1] \vee \{ (x=2) \wedge (y \neq 0) \} && \text{quantifier neg. law} \\
 \Leftrightarrow & (\exists x) (\forall y) [\neg \{ f(x,y)=1 \} \wedge \neg \{ (x=2) \wedge (y \neq 0) \}] && \text{De Morgan's laws} \\
 \Leftrightarrow & (\exists x) (\forall y) [\{ f(x,y) \neq 1 \} \wedge \{ (x \neq 2) \vee y=0 \}] && "
 \end{aligned}$$

3(a) $(a,b) = \{ \{a\}, \{a,b\} \}$.

(b) Suppose $(a,b) \in (A-B) \times C$. Then $a \in A-B$ and $b \in C$. So $a \in A$ & $a \notin B$ and $b \in C$. $\therefore a \in A \wedge b \in C$ and $a \notin B \wedge b \in C$. $\therefore (a,b) \in A \times C$ and $(a,b) \notin B \times C$. [As soon as $a \notin B$, $(a,b) \notin B \times C$] $\therefore (a,b) \in (A \times C) - (B \times C)$. So $(A-B) \times C \subseteq (A \times C) - (B \times C)$... (1)

Now suppose $(a,b) \in (A \times B) - (B \times C)$. Then $(a,b) \in A \times C$ and $(a,b) \notin B \times C$. So $a \in A$ & $b \in C$. Since $b \in C$, the only for us to have $(a,b) \notin B \times C$ is for $a \notin B$. $\therefore a \in A-B$ and $b \in C$. So $(a,b) \in (A-B) \times C$. Thus $(A \times C) - (B \times C) \subseteq (A-B) \times C$ (2). From (1) & (2) we get $(A-B) \times C = (A \times C) - (B \times C)$.

$$4(a) \quad R \circ S = \{(a, c) : (\exists b) [(a, b) \in S \wedge (b, c) \in R]\}$$

(b) For each $a \in \mathbb{Z}$, we have $a^3 - a^3 = 0 = 8(0)$. So $(\forall a \in \mathbb{Z})[aRa]$.

$\therefore R$ is reflexive. Now suppose for $a, b \in \mathbb{Z}$, aRb . Then $a^3 - b^3 = 8k$ with $k \in \mathbb{Z}$. So $b^3 - a^3 = -(8k) = 8(-k)$. Since $-k \in \mathbb{Z}$, bRa . Thus $(\forall a, b \in \mathbb{Z})[aRb \rightarrow bRa]$. $\therefore R$ is symmetric.

Finally suppose for $a, b, c \in \mathbb{Z}$, we have $aRb \wedge bRc$. Then $a^3 - b^3 = 8k$ & $b^3 - c^3 = 8l$ with $k, l \in \mathbb{Z}$. So $a^3 - c^3 = (a^3 - b^3) + (b^3 - c^3) = 8k + 8l = 8(k+l)$. Since $k+l \in \mathbb{Z}$, aRc . So $(\forall a, b, c \in \mathbb{Z})[aRb \wedge bRc \rightarrow aRc]$. $\therefore R$ is transitive. Hence R is an equivalence relation.

$$(c) \quad 1^3 \equiv 1 \pmod{8}, \quad 2^3 = 8 \equiv 0 \pmod{8}, \quad 3^3 = 27 \equiv 3 \pmod{8}$$

$$4^3 = 64 \equiv 0 \pmod{8}, \quad 5^3 = (25)(5) \equiv (1)(5) \equiv 5 \pmod{8}$$

$$6^3 = (36)(6) \equiv 4(6) \equiv 24 \equiv 0 \pmod{8}, \quad 7^3 = (49)(7) \equiv (1)(7) \equiv 7 \pmod{8}$$

Thus $[0]_R = [2]_R = [4]_R = [6]_R$ and $[0]_R, [1]_R, [3]_R, [5]_R$ & $[7]_R$ are all different. Hence R partitions \mathbb{Z} into the equiv. classes

$$[0]_R = \{8k : k \in \mathbb{Z}\} \cup \{8k+2 : k \in \mathbb{Z}\} \cup \{8k+4 : k \in \mathbb{Z}\} = \{2k : k \in \mathbb{Z}\}$$

$$[1]_R = \{8k+1 : k \in \mathbb{Z}\}, \quad [3]_R = \{8k+3 : k \in \mathbb{Z}\}$$

$$[5]_R = \{8k+5 : k \in \mathbb{Z}\}, \text{ and } [7]_R = \{8k+7 : k \in \mathbb{Z}\}.$$

5(a) $f: N \rightarrow Q$ is injective if $(\forall x_1, x_2 \in N) [f(x_1) = f(x_2) \rightarrow (x_1 = x_2)]$

$f: N \rightarrow Q$ is surjective if $(\forall y \in Q) (\exists x \in N) [f(x) = y]$

(b) We will first show that $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{5\}$ is injective.

So let $x_1, x_2 \in \mathbb{R} - \{3\}$ and suppose $f(x_1) = f(x_2)$. Then

$$(5x_1)/(x_1-3) = (5x_2)/(x_2-3)$$

$$\text{So } (5x_1)(x_2-3) = (5x_2)(x_1-3) \Rightarrow 5x_1x_2 - 15x_1 = 5x_1x_2 - 15x_2$$

$$\therefore -15x_1 = -15x_2 \text{. So } x_1 = x_2 \text{. Hence}$$

$(\forall x_1, x_2 \in \mathbb{R} - \{3\}) [f(x_1) = f(x_2) \rightarrow (x_1 = x_2)]$. So $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{5\}$ is injective.

5(c) We will now show that $f: R - \{3\} \rightarrow R - \{5\}$ is surjective.

Let $y \in R - \{5\}$ be given. Choose $x = 3y/(y-5)$. Then

$$x = \frac{3y}{y-5} = \frac{3(y-5)+15}{y-5} = 3 + \frac{15}{y-5} \in R - \{3\} \text{ because}$$

$15/(y-5)$ can never be 0. Also $f(x) = 5x/(x-3)$

$$= \frac{5(x-3)+15}{x-3} = 5 + \frac{15}{x-3} = 5 + \frac{15}{15/(y-5)} \text{ b.c. } x = 3 + \frac{15}{y-5}$$

$$= 5 + \frac{15}{15} \cdot (y-5) = 5 + (y-5) = y. \therefore f(x) = y.$$

So $(\forall y \in R - \{5\}) (\exists x \in R - \{3\}) [f(x) = y]$. $\therefore f$ is surjective.

Hence f is both injective & surjective. So it is bijective.

6(a) $\bigcap_{i \in I} A_i = \{x : (\forall i \in I)(x \in A_i)\}$, $\bigcup_{i \in I} A_i = \{x : (\exists i \in I)(x \in A_i)\}$

(b) Suppose $x \in (B - \bigcup_{i \in I} A_i)$. Then $x \in B$ and $x \notin \bigcup_{i \in I} A_i$. So

$x \in B$ and for all $i \in I$, $x \notin A_i$. So for all $i \in I$, $x \in (B - A_i)$

$$\therefore x \in \bigcap_{i \in I} (B - A_i). \text{ Hence } (B - \bigcup_{i \in I} A_i) \subseteq \bigcap_{i \in I} (B - A_i) \dots (1)$$

Now suppose $x \in \bigcap_{i \in I} (B - A_i)$. Then for all $i \in I$, $x \in (B - A_i)$.

So for all $i \in I$, $x \in B \wedge x \notin A_i$. $\therefore x \in B$ and for all $i \in I$, $x \notin A_i$. $\therefore x \in B$ and $x \notin \bigcup_{i \in I} A_i$. So

$$x \in B - \bigcup_{i \in I} A_i. \text{ Hence } \bigcap_{i \in I} (B - A_i) \subseteq (B - \bigcup_{i \in I} A_i) \dots (2)$$

From (1) & (2) it follows that $B - \bigcup_{i \in I} A_i = \bigcap_{i \in I} (B - A_i)$

Alternative solution to 6(b) :

$$\begin{aligned} x \in (B - \bigcup_{i \in I} A_i) &\Leftrightarrow x \in B \wedge x \notin \bigcup_{i \in I} A_i \Leftrightarrow (x \in B) \wedge \neg \left(\bigcup_{i \in I} (x \in A_i) \right) \\ &\Leftrightarrow x \in B \wedge \neg (\exists i \in I) (x \in A_i) \\ &\Leftrightarrow x \in B \wedge (\forall i \in I) [\neg (x \in A_i)] \\ &\Leftrightarrow (x \in B) \wedge (\forall i \in I) (x \notin A_i) \\ &\Leftrightarrow (\forall i \in I) (x \in B \wedge x \notin A_i) \\ &\Leftrightarrow (\forall i \in I) [x \in (B - A_i)] \Leftrightarrow x \in \bigcap_{i \in I} (B - A_i) \end{aligned}$$

$$\therefore B - \bigcup_{i \in I} A_i = \bigcap_{i \in I} (B - A_i).$$