

Answer all 6 questions. **No Calculators or Cell phones are allowed.** An unjustified answer will receive little or no credit. **BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.**

- (15) 1(a) Define what is a *finite sequence* and define what is an *infinite sequence*.  
(b) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the function with  $f(x) = x/(x-3)$ , if  $x \neq 3$ ; and  $f(3) = 1$ . Find  $f^{-1}(x)$  for each  $x \in \mathbb{R}$ .
- (20) 2(a) Let  $f : X \rightarrow Y$  be a function and suppose that  $A, B \subseteq X$  and  $C, D \subseteq Y$ . Define what is  $f[A]$  and what is  $f^{-1}[C]$ .  
(b) Is it always true that  $f[A] \cap f[B] \subseteq f[A \cap B]$ ?  
(c) Is it always true that  $f^{-1}[C] - f^{-1}[D] \subseteq f^{-1}[C - D]$ ?
- (15) 3(a) Write down what the *First Principle of Mathematical Induction* says.  
(b) Prove that  $(\forall n \in \mathbb{N})(3^{n+2} + 4^{2n+1}$  is divisible by 13).
- (20) 4(a) Define exactly when  $A$  is a *finite set* & exactly when  $A$  is a *denumerable set*.  
(b) Prove that  $\mathbb{N} \times \mathbb{N}$  is denumerable. [If you claim that a function is a bijection, you must prove that this is indeed so.]
- (15) 5(a) Define what it means for the infinite sequence  $\langle a_n \rangle_{n \in \mathbb{N}}$  to be *convergent*.  
(b) Suppose  $\langle a_n \rangle_{n \in \mathbb{N}}$  converges to  $A$  and  $\langle b_n \rangle_{n \in \mathbb{N}}$  converges to  $B$ . Prove that  $\langle a_n - b_n \rangle_{n \in \mathbb{N}}$  converges to  $A - B$ .
- (15) 6(a) Define what it means for the sequence  $\langle a_n \rangle_{n \in \mathbb{N}}$  to be a *Cauchy sequence*.  
(b) Prove that if  $\langle a_n \rangle_{n \in \mathbb{N}}$  is a Cauchy sequence, then it is *bounded above* & *bounded below*.

1(a) A finite sequence is a function with domain  $\mathbb{N}_k = \{0, 1, \dots, k-1\}$  for some  $k \in \mathbb{N}$ . An infinite seq. is a function with domain  $\mathbb{N}$ .

(b) Let  $y = f(x)$  and suppose  $x \neq 3$ . Then  $y \neq 1$  &  $f^{-1}(y) = x$ . Since  $f(x) = x/(x-3)$ ,  $y = f(x) = x/(x-3)$ .  $\therefore y(x-3) = x$ . So  $yx - 3y = x$   
 $\therefore x(y-1) = 3y$ . So  $x = 3y/(y-1)$ .  $\therefore f^{-1}(y) = x = 3y/(y-1)$  for  $y \neq 1$ . Also  $f^{-1}(1) = 3$ . So  $f^{-1}(x) = \begin{cases} 3x/(x-1) & \text{if } x \neq 1, \\ 3 & \text{if } x = 1. \end{cases}$

2(a)  $f[A] = \{f(x) : x \in A\}$  &  $f^{-1}[C] = \{x \in X : f(x) \in C\}$ .

(b) NO. Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(k) = k^2$ . Take  $A = \{-2, 0\}$  and  $B = \{0, 2\}$ . Then  $f[A] = \{f(-2), f(0)\} = \{0, 4\}$  and  $f[B] = \{f(0), f(2)\} = \{0, 4\}$ . Also  $A \cap B = \{0\}$ , so  $f[A \cap B] = \{f(0)\} = \{0\}$ . Now  $f[A] \cap f[B] = \{0, 4\} \neq \{0\} = f[A \cap B]$ . So it is not always true that  $f[A] \cap f[B] \subseteq f[A \cap B]$ .

(c) YES. Let  $x \in f^{-1}[C] - f^{-1}[D]$ . Then  $x \in f^{-1}[C]$  &  $x \notin f^{-1}[D]$ . So  $f(x) \in C$  &  $f(x) \notin D$  (bec. if  $f(x) \in D$ , then we would have  $x \in f^{-1}[D]$ ). Hence  $f(x) \in C - D$ . So  $x \in f^{-1}[C - D]$ .  $\therefore f^{-1}[C] - f^{-1}[D] \subseteq f^{-1}[C - D]$  for all  $C, D$ .

3(a) Let  $P(n)$  be a formula of first order logic with free variable  $n$ . Then  $\{P(0) \wedge (\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]\} \Rightarrow \{(\forall n \in \mathbb{N}) P(n)\}$ .

(b)  $3^{0+2} + 4^{2(0)+1} = 9 + 4 = 13$ . So  $P(0)$  is true. Suppose  $P(n)$  is true. Then  $3^{n+2} + 4^{2n+1} = 13k$  for some  $k \in \mathbb{Z}$ . Now  
 $3^{(n+1)+2} + 4^{2(n+1)+1} = 3 \cdot 3^{n+2} + 16 \cdot 4^{2n+1} = 3 \cdot (3^{n+2} + 4^{2n+1}) + 13 \cdot 4^{2n+1}$   
 $= 3(13k) + 13 \cdot 4^{2n+1} = 13(3k + 4^{2n+1})$ .

So  $P(n+1)$  is true.  $\therefore P(0) \wedge (\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]$ . Hence  $(\forall n \in \mathbb{N}) P(n)$ . So  $(\forall n \in \mathbb{N}) (3^{n+2} + 4^{2n+1})$  is divisible by 13.

Let  $P(n)$  be the statement:  
 $3^{n+2} + 4^{2n+1}$  is divisible by 13.  
 Then ...

4(a) The set  $A$  is finite if we can find a bijection from  $A$  to  $\mathbb{N}_k = \{0, 1, 2, \dots, k-1\}$  for some  $k \in \mathbb{N}$ .  $A$  is denumerable if we can find a bijection from  $A$  to  $\mathbb{N}$ .

(b) Define  $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  by  $f(k, l) = 2^k(2l+1) - 1$ . Now suppose  $f(k_1, l_1) = f(k_2, l_2)$ . Then  $2^{k_1}(2l_1+1) - 1 = 2^{k_2}(2l_2+1) - 1$ . So  $2^{k_1}(2l_1+1) = 2^{k_2}(2l_2+1)$ . Since  $2l_1+1$  &  $2l_2+1$  are odd, we must have  $2^{k_1} = 2^{k_2}$ , so  $k_1 = k_2$ . Thus  $2l_1+1 = 2l_2+1$ . So  $l_1 = l_2$ . Hence  $(k_1, l_1) = (k_2, l_2)$ . So  $f$  is injective. Let  $b \in \mathbb{N}$ . Then  $b+1$  can be expressed in the form  $2^k(2l+1)$  with  $k, l \in \mathbb{N}$ . Now  $f(k, l) = 2^k(2l+1) - 1 = (b+1) - 1 = b$ . So  $f$  is surjective. Hence  $f$  is a bijection.  $\therefore \mathbb{N} \times \mathbb{N}$  is denumerable.

5(a)  $\langle a_n \rangle_{n \in \mathbb{N}}$  is convergent if  $(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|a_n - L| < \epsilon)$

(b) Let  $\epsilon > 0$  be given. Then  $\epsilon/2 > 0$ . Since  $\langle a_n \rangle$  conv. to  $A$  &  $\langle b_n \rangle$  conv. to  $B$  we can find  $N_1, N_2 \in \mathbb{N}$  such that  $(\forall n \geq N_1)(|a_n - A| < \epsilon/2)$  &  $(\forall n \geq N_2)(|b_n - B| < \epsilon/2)$ .

Let  $N = \max\{N_1, N_2\}$ . Then  $\forall n \geq N$  we have

$$\begin{aligned} |(a_n - b_n) - (A - B)| &= |a_n - A + B - b_n| \leq |a_n - A| + |B - b_n| \\ &= |a_n - A| + |b_n - B| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

$\therefore (\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|(a_n - b_n) - (A - B)| < \epsilon)$ .  $\therefore \langle a_n - b_n \rangle$  conv. to  $A - B$ .

6(a)  $\langle a_n \rangle_{n \in \mathbb{N}}$  is a Cauchy seq. if  $(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)(|a_m - a_n| < \epsilon)$ .

(b) Let  $\epsilon = 1$ . Since  $\langle a_n \rangle$  is a Cauchy seq.  $(\exists N \in \mathbb{N})(\forall m, n \geq N)(|a_m - a_n| < 1)$ . So in particular  $\forall m \geq N$ ,  $(|a_m - a_N| < 1)$ , i.e.  $a_N - 1 < a_m < a_N + 1$ .

Let  $U = \max\{a_0, a_1, a_2, \dots, a_{N-1}, a_N + 1\}$  and  $L = \min\{a_0, a_1, a_2, \dots, a_{N-1}, a_N - 1\}$ . Then  $(\forall m \in \mathbb{N})(a_m \leq U)$  and  $(\forall m \in \mathbb{N})(L \leq a_m)$ . So  $\langle a_n \rangle$  is bounded above by  $U$  and bounded below by  $L$ . END