

Answer all 6 questions. No calculators or Cell phones are allowed. An unjustified answer will receive little credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (15) 1(a) Translate the following argument into *symbolic language*.
"Either Abe or Dan will win. If Dan wins, then Ben will not win.
Therefore, if Ben won then Abe did not win."
(b) Use a *truth table* to determine if this argument is *logically valid*.
- (15) 2(a) Define $(\exists x \in B)P(x)$ and $(\forall x \in B)P(x)$ in terms of *unbounded quantifiers*.
(b) Prove that $\neg[(\exists x \in B)P(x)]$ is *logically equivalent* to $(\forall x \in B)[\neg P(x)]$.
(Justify each step of your argument.)
- (15) 3(a) Define what is a *function* in terms of ordered pairs.
(b) Prove that $A \times (B - C) = (A \times B) - (A \times C)$ for all sets A, B, and C.
- (20) 4(a) Let R & S be *relations*. Define what are the relations R^{-1} and $S \circ R$.
(b) Let R be the relation on \mathbb{Z} defined by: aRb if $(b^3 - a^3)$ is an *integer multiple* of 6. Prove that R is an *equivalence relation* and find the *equivalence classes* into which R partitions \mathbb{Z} . (Specify each part of the partition.)
- (20) 5(a) Define what it means for the function $f: \mathbb{Q} \rightarrow \mathbb{N}$ to be *injective* and what it means for it to be *surjective*.
(b) Let $f: \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{3\}$ be the function with $f(x) = (3x+1) / (x-2)$. Prove that f is a *bijection*.
- (15) 6(a) Let $\langle A_i : i \in I \rangle$ be an *indexed family* of subsets of a universal set U. Define what are $\bigcap_{i \in I} (A_i)$ and $\bigcup_{i \in I} (A_i)$.
(b) Prove that $[\bigcap_{i \in I} (A_i)]^c = \bigcup_{i \in I} (A_i^c)$.

1(a) Let $A = \text{Abe wins}$, $B = \text{Ben wins}$, and $D = \text{Dan wins}$. Argument says $[(A \vee D) \wedge (D \rightarrow \neg B)] \Rightarrow (B \rightarrow \neg A)$.

A	B	D	$[(A \vee D) \wedge (D \rightarrow \neg B)]$	$\rightarrow (B \rightarrow \neg A)$
T	T	T	F	F
T	T	F	T	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	F	T

Since $[(A \vee D) \wedge (D \rightarrow \neg B)] \rightarrow (B \rightarrow \neg A)$ is not a tautology
 the argument is not logically valid.

2(a) $(\exists x \in B) P(x)$ is defined to be $(\exists x)[(x \in B) \wedge P(x)]$

$(\forall x \in B) P(x)$ is defined to be $(\forall x)[(x \in B) \rightarrow P(x)]$

$$\begin{aligned}
 \text{(b)} \quad \neg[(\exists x \in B) P(x)] &\Leftrightarrow \neg(\exists x)[(x \in B) \wedge P(x)] \text{ from the definition in 2(a)} \\
 &\Leftrightarrow (\forall x) \neg[(x \in B) \wedge P(x)] \text{ by Quantifier negation law} \\
 &\Leftrightarrow (\forall x) [\neg(x \in B) \vee \neg P(x)] \text{ by De Morgan's law} \\
 &\Leftrightarrow (\forall x) [(x \in B) \rightarrow \neg P(x)] \text{ b.c. } (R \rightarrow Q) \Leftrightarrow (\neg R \vee Q) \\
 &\Leftrightarrow (\forall x \in B) [\neg P(x)] \text{ from the definition in 2(a)}
 \end{aligned}$$

3 (a) A function is a set of ordered pairs, f , such that

$$[(a, b) \in f \wedge (a, c) \in f] \rightarrow b = c.$$

(b) Suppose $(a, b) \in A \times (B - C)$. Then $a \in A$ and $b \in B - C$, so $a \in A$ and $b \in B$ and $b \notin C$. $\therefore a \in A \wedge b \in B$ and $a \in A \wedge b \notin C$. Hence $(a, b) \in A \times B$ and $(a, b) \notin A \times C$. So $(a, b) \in A \times B - A \times C$. Thus $A \times (B - C) \subseteq A \times B - A \times C \dots \textcircled{1}$. Now suppose that $(a, b) \in A \times B - A \times C$.

Then $(a, b) \in A \times B$ and $(a, b) \notin A \times C$. So $a \in A \wedge b \in B$ and $\neg(a \in A \wedge b \in C)$

$\therefore a \in A \wedge b \in B$ and $\neg(a \in A) \vee \neg(b \in C)$. Since $a \in A$, $\neg(a \in A)$ is false, so $b \notin C$. $\therefore a \in A$ and $b \in B \wedge b \notin C$. $\therefore a \in A \wedge b \in (B - C)$. Hence $(a, b) \in A \times (B - C)$. Thus $A \times B - A \times C \subseteq A \times (B - C) \dots \textcircled{2}$. From $\textcircled{1}$ and $\textcircled{2}$ it follows that $A \times (B - C) = A \times B - A \times C$.

- 4(a) $R^{-1} = \{(b, a) : (a, b) \in R\}$, so $R = \{(a, c) : (\exists b)[(a, b) \in R \wedge (b, c) \in S]\}$
- (b) Since $a^3 - a^3 = 6(0)$, $(\forall a \in \mathbb{Z})[aRa]$. $\therefore R$ is reflexive on \mathbb{Z} .
- Suppose aRb . Then $b^3 - a^3 = 6k$ for some $k \in \mathbb{Z}$. So $a^3 - b^3 = 6(-k)$ and thus bRa . $\therefore (\forall a, b \in \mathbb{Z})[aRb \rightarrow bRa]$. $\therefore R$ is symmetric on \mathbb{Z} .
- Suppose $aRb \wedge bRc$. Then $b^3 - a^3 = 6k$ and $c^3 - b^3 = 6l$ for some $k, l \in \mathbb{Z}$. $\therefore c^3 - a^3 = (c^3 - b^3) + (b^3 - a^3) = 6(l+k)$. $\therefore aRc$. $\therefore (\forall a, b, c \in \mathbb{Z})[aRb \wedge bRc \rightarrow aRc]$. $\therefore R$ is transitive on \mathbb{Z} .
- Hence R is an equivalence relation.
- (c) $0^3 \equiv 0 \pmod{6}$, $1^3 \equiv 1 \pmod{6}$, $2^3 \equiv 2 \pmod{6}$, $3^3 \equiv 3 \pmod{6}$,
 $4^3 \equiv 4 \pmod{6}$, $5^3 \equiv 5 \pmod{6}$. So $[0]_R, [1]_R, [2]_R, [3]_R, [4]_R, [5]_R$ are all different. Hence the equivalence classes are
 $[0]_R = \{6k : k \in \mathbb{Z}\}$, $[1]_R = \{6k+1 : k \in \mathbb{Z}\}$, $[2]_R = \{6k+2 : k \in \mathbb{Z}\}$
 $[3]_R = \{6k+3 : k \in \mathbb{Z}\}$, $[4]_R = \{6k+4 : k \in \mathbb{Z}\}$, & $[5]_R = \{6k+5 : k \in \mathbb{Z}\}$.

- 5(a) $f: \mathbb{Q} \rightarrow \mathbb{N}$ is injective if $(\forall x_1, x_2 \in \mathbb{Q})[x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)]$
This is equivalent to saying $(\forall x_1, x_2 \in \mathbb{Q})[f(x_1) = f(x_2) \rightarrow x_1 = x_2]$
 $f: \mathbb{Q} \rightarrow \mathbb{N}$ is surjective if $(\forall y \in \mathbb{N})(\exists x \in \mathbb{Q})[f(x) = y]$
- (b) Suppose $f(x_1) = f(x_2)$. Then $\frac{3x_1 + 1}{x_1 - 2} = \frac{3x_2 + 1}{x_2 - 2}$
So $3x_1 x_2 - 6x_1 + x_2 - 2 = 3x_1 x_2 + x_1 - 6x_2 - 2$.
 $\therefore -7x_1 = -7x_2$. So $x_1 = x_2$. Hence f is injective.

- (c) Let $y \in \mathbb{R} - \{3\}$. Put $x = 2 + \frac{7}{y-3}$. [We get this x by solving the equation $(3x+1)/(x-2) = y$ for x in terms of y]
Then x is well-defined because y is never 3 and also $x \in \mathbb{R} - \{2\}$ because $7/(y-3)$ can never be 0. Finally,

$$\begin{aligned} f(x) &= f\left(2 + \frac{7}{y-3}\right) = \frac{3\left(2 + \frac{7}{y-3}\right) + 1}{2 + \frac{7}{y-3} - 2} = \frac{7 + \frac{21}{y-3}}{\frac{7}{y-3}} \\ &= \frac{(7y - 21 + 21)/(y-3)}{7/(y-3)} = \frac{7y}{7} \cdot \frac{y-3}{y-3} = y. \end{aligned}$$

Hence we have found an $x \in \mathbb{R} - \{2\}$ with $f(x) = y$.
Thus f is surjective. $\therefore f$ is a bijection.

$$6(a) \quad \bigcap_{i \in I} (A_i) = \{x \in U : (\forall i \in I)(x \in A_i)\}$$

$$\bigcup_{i \in I} (A_i) = \{x \in U : (\exists i \in I)(x \in A_i)\}$$

(b) Suppose $x \in [\bigcap_{i \in I} (A_i)]^c$. Then $x \in U - (\bigcap_{i \in I} A_i)$. So $x \in U \wedge x \notin \bigcap_{i \in I} (A_i)$. $\therefore x \in U$ and there is an $i_0 \in I$ such that $x \notin A_{i_0}$. Thus $x \in U - A_{i_0} = A_{i_0}^c$. $\therefore x \in \bigcup_{i \in I} (A_i^c)$. Hence $[\bigcap_{i \in I} (A_i)]^c \subseteq \bigcup_{i \in I} (A_i^c) \dots (1)$

Now suppose $x \in \bigcup_{i \in I} (A_i^c)$. Then for some $i_0 \in I$ $x \in A_{i_0}^c$. So $x \in U - A_{i_0}^c$. $\therefore x \in U$ and $x \notin A_{i_0}$. $\therefore x \in U$ and $\neg[(\forall i \in I)(x \in A_i)]$. So $x \in U$ and $x \notin \bigcap_{i \in I} (A_i)$. $\therefore x \in U - \bigcap_{i \in I} (A_i) = [\bigcap_{i \in I} (A_i)]^c$. Hence $\bigcup_{i \in I} (A_i^c) \subseteq [\bigcap_{i \in I} (A_i)]^c \dots (2)$. From (1) and (2) it follows that $[\bigcap_{i \in I} (A_i)]^c = \bigcup_{i \in I} (A_i^c)$

END.