

*Answer all 6 questions. No Calculators or Cell phones are allowed. An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.*

- (15) 1(a) Define what is an *infinite sequence* and define what is a *finite sequence*.  
(b) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function with  $f(x) = 2x/(x-1)$ , if  $x \neq 1$ ; and  $f(1) = 2$ . Find (with justification)  $f^{-1}(x)$  for each  $x \in \mathbb{R}$ .
- (19) 2(a) Let  $f: X \rightarrow Y$  be a function and suppose that  $A, B \subseteq X$  and  $C, D \subseteq Y$ . Define what is  $f[A]$  and what is  $f^{-1}[C]$ .  
(b) Is it always true that  $f[A \cup B] \subseteq f[A] \cup f[B]$ ?  
(c) Is it always true that  $f^{-1}[C \cap D] \subseteq f^{-1}[C] \cap f^{-1}[D]$ ?
- (18) 3(a) Write down what is the *Second (Strong) Principle of Mathematical Induction*.  
(b) Prove that (i)  $(\forall n \in \mathbb{N})(n+1 \leq 2^n)$  and (ii)  $(\forall n \in \mathbb{N})(n! \leq 2^{n,n})$ .
- (18) 4(a) Define what it means for A to be a *finite set* & for B to be a *denumerable set*.  
(b) Prove that  $\mathbb{Z} \times \mathbb{Z}$  is denumerable. [If you claim that a function is a bijection, you must prove that this is indeed so.]
- (15) 5(a) Define what it means for the infinite sequence  $\langle a_n \rangle_{n \in \mathbb{N}}$  to be *convergent*.  
(b) Suppose  $\langle b_n \rangle_{n \in \mathbb{N}}$  converges to B,  $b_n$  is never 0, and  $B \neq 0$ . Prove that  $(\exists N \in \mathbb{N}) (\forall n \geq N) [ |b_n| > |B|/2 ]$ .
- (15) 6(a) Define what it means for the sequence  $\langle a_n \rangle_{n \in \mathbb{N}}$  to be a *Cauchy sequence*.  
(b) Prove that if  $\langle a_n \rangle_{n \in \mathbb{N}}$  is a Cauchy sequence and  $c \neq 0$ , then  $\langle a_n/c \rangle_{n \in \mathbb{N}}$  is also a Cauchy sequence.

- 1(a) A finite sequence is a function with domain  $\mathbb{N}_k = \{0, 1, 2, \dots, k-1\}$  for some  $k \in \mathbb{N}$ . An inf. sequence is a function with domain  $\mathbb{N}$
- (b) Let  $y = f(x)$  and suppose  $x \neq 1$ . Then  $y \neq 2$  &  $f^{-1}(x) = y$ . Since  $f(x) = 2x/(x-1)$ ,  $y = f(x) = 2x/(x-1)$ .  $\therefore y(x-1) = 2x$ . So  $yx - y = 2x$ .  
 $\therefore x(y-2) = y$  So  $x = y/(y-2)$ .  $\therefore f^{-1}(y) = x = y/(y-2)$ , if  $y \neq 2$   
Also  $f^{-1}(2) = 1$ . Hence  $f^{-1}(x) = \begin{cases} x/(x-2) & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ .

2(a)  $f[A] = \{y \in Y : (\exists x \in A)[y = f(x)]\}$        $f^{-1}[C] = \{x \in X : f(x) \in C\}$ .

(b) YES. Let  $y \in f[A \cup B]$ . Then we can find an  $x \in A \cup B$  such that  $y = f(x)$ . Since  $x \in A \cup B$ ,  $x \in A$  or  $x \in B$ . Now if  $x \in A$ , then  $y = f(x) \wedge x \in A$ , so  $y \in f[A]$ . And if  $x \in B$ , then  $y = f(x) \wedge x \in B$ , so  $y \in f[B]$ .  $\therefore y \in f[A]$  or  $y \in f[B]$   
 $\therefore y \in f[A] \cup f[B]$ . Hence  $f[A \cup B] \subseteq f[A] \cup f[B]$

(c) YES. Let  $x \in f^{-1}[C-D]$ . Then  $f(x) \in C-D$ , so  $f(x) \in C$  and  $f(x) \notin D$ . Thus  $x \in f^{-1}[C]$  and  $x \notin f^{-1}[D]$  (bec. if  $x \in f^{-1}[D]$  then we would have  $f(x) \in D$ ).  $\therefore x \in f^{-1}[C] - f^{-1}[D]$ .  
Hence  $f^{-1}[C-D] \subseteq f^{-1}[C] - f^{-1}[D]$ .

- 3(a) Let  $P(n)$  be a formula of first order logic with free variable  $n$ .

$$\text{Then } \{P(0) \wedge (\forall n \in \mathbb{N})[P(0) \wedge P(1) \wedge \dots \wedge P(n) \rightarrow P(n+1)]\} \Rightarrow (\forall n \in \mathbb{N})[P(n)].$$

(b) Let  $P(n)$  be the formula,  $[(n+1) \leq 2^n]$ . Since  $0+1 \leq 1 = 2^0$ ,  $P(0)$  is true. Now suppose  $P(n)$  is true. Then  $(n+1) \leq 2^n$ . So  $(n+1)+1 \leq (n+1)+(n+1) = 2(n+1) \leq 2 \cdot 2^n = 2^{n+1}$ . So  $P(n+1)$  is true.

Hence  $(\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]$ .  $\therefore$  by the First Principle of Math Induction  $(\forall n \in \mathbb{N})[P(n)]$ . Hence  $(\forall n \in \mathbb{N})[(n+1) \leq 2^n]$ .

(c) Let  $Q(n)$  be the formula  $[n! \leq 2^{n^2}]$ . Since  $0! = 1 \leq 2^{0^2}$ ,  $Q(0)$  is true. Now suppose  $Q(n)$  is true. Then  $n! \leq 2^{n^2}$ . So

3(c)  $(n+1)! = (n+1) \cdot n! \leq 2^n \cdot 2^{n^2} = 2^{n^2+n} \leq 2^{n^2+2n+1} = 2^{(n+1)^2}$ , by using the fact  $(n+1) \leq 2^n$  from part (b). Thus  $Q(n+1)$  is true. Hence  $(\forall n \in \mathbb{N})[Q(n) \rightarrow Q(n+1)]$ . i.e., by the first Principle of Math Induction,  $(\forall n \in \mathbb{N})[Q(n)]$ . Hence  $(\forall n \in \mathbb{N})[n! \leq 2^{n^2}]$ .

4(a) The set  $A$  is finite if  $A \approx \mathbb{N}_k$  for some  $k \in \mathbb{N}$ .  $A$  is denumerable if  $A \approx \mathbb{N}$ , i.e., there is a bijection from  $A$  to  $\mathbb{N}$ .

(b) Let  $g: \mathbb{Z} \rightarrow \mathbb{N}$  be defined by  $g(n) = \begin{cases} 2^n & \text{if } n \geq 0 \\ -(2n+1) & \text{if } n < 0 \end{cases}$ . Then  $g$  is a bijection because  $g$  takes  $\mathbb{Z}^+$  onto the even natural numbers and  $g$  takes  $\mathbb{Z}^-$  onto the odd natural numbers and the correspondence is one-to-one. Now define  $f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{N}$  by  $f(k, l) = 2^{g(k)} \cdot [2 \cdot g(l) + 1] - 1$ . We claim that  $f$  is a bijection. This will show that  $\mathbb{Z} \times \mathbb{Z}$  is denumerable.

Suppose  $f(k_1, l_1) = f(k_2, l_2)$ . Then  $2^{g(k_1)} \cdot [2 \cdot g(l_1) + 1] - 1 = 2^{g(k_2)} \cdot [2 \cdot g(l_2) + 1] - 1$ . So  $2^{g(k_1)} \cdot [2 \cdot g(l_1) + 1] = 2^{g(k_2)} \cdot [2 \cdot g(l_2) + 1]$   
 $\therefore 2^{g(k_1)} = 2^{g(k_2)}$  and  $2 \cdot g(l_1) + 1 = 2 \cdot g(l_2) + 1$ . So  $g(k_1) = g(k_2)$

and  $g(l_1) = g(l_2)$ .  $\therefore k_1 = k_2$  &  $l_1 = l_2$  b.c.  $g$  is bijective.

Hence  $f$  is injective. Let  $n \in \mathbb{N}$ . Then we can write  $n+1$  in the form  $2^p \cdot (2q+1)$  with  $p, q \in \mathbb{N}$ . Put  $k = \tilde{g}'(p)$  and  $l = \tilde{g}'(q)$ . Then  $f(k, l) = 2^{g(k)} \cdot [2 \cdot g(l) + 1] - 1 = 2^p \cdot (2q+1) - 1 = (n+1) - 1 = n$ . So  $f$  is surjective. Hence  $f$  is a bijection.

5(a)  $\langle a_n \rangle$  is convergent if  $(\exists L \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|a_n - L| < \varepsilon]$ .

(b) Let  $\varepsilon = |B|/2$ . Then  $\varepsilon > 0$ . So we can find  $a_N \in \mathbb{N}$  such that  $\forall n \geq N$   $|b_n - B| < \varepsilon$ . Hence  $\forall n \geq N$  we have  $|b_n| = |B - (B - b_n)| \geq |B| - |B - b_n| = |B| - |b_n - B| > |B| - \varepsilon = |B| - |B|/2 = |B|/2$ .  $\therefore (\exists N \in \mathbb{N})(\forall n \geq N)[|b_n| > |B|/2]$ .

6(a)  $\langle a_n \rangle_{n \in \mathbb{N}}$  is a Cauchy seq. if  $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)[|a_m - a_n| < \varepsilon]$ .

(b) Let  $\varepsilon > 0$  be given. Then  $|c|, \varepsilon > 0$ . So we can find an  $N \in \mathbb{N}$  s.t.  $\forall m, n \geq N$   $|a_m - a_n| < |c| \cdot \varepsilon$ . Hence  $\forall m, n \geq N$ ,  $\left| \frac{a_m}{c} - \frac{a_n}{c} \right| = \frac{1}{|c|} \cdot |a_m - a_n| < \frac{1}{|c|} \cdot |c| \cdot \varepsilon = \varepsilon$ . Hence  $\langle a_n/c \rangle$  is also a Cauchy sequence.