

Answer all 6 questions. No calculators or Cell phones are allowed. An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (15) 1(a) Translate the following argument into *symbolic language*.
"If Adam is guilty then Chad is not guilty. Either Adam or Ben is guilty.
Therefore, if Chad is guilty then Ben is guilty."
(b) Use a *truth table* to determine if this argument is *logically valid*.
- (15) 2(a) Define $(\exists !x)[Q(x)]$ and $(\forall x \in A)[R(x)]$ in terms of *unbounded quantifiers*.
(b) Convert the formula $\neg(\forall x)(\exists y) [\{f(x) = f(y)\} \rightarrow \{\neg(x = y) \wedge (x+y = 0)\}]$ into a *logically equivalent formula* in which no " \neg " sign governs a *quantifier* or a *connective*. [Specify which law you use at each step.]
- (20) 3(a) Let R & S be *relations*. Define what are the relations R^{-1} and $S \circ R$.
(b) Let R be the relation on \mathbb{Z} defined by aRb if $(a^2 - b^2)$ is an *integer multiple* of 10. Prove that R is an *equivalence relation* and find the *equivalence classes* into which R partitions \mathbb{Z} . (Specify each part of the partition.)
- (15) 4(a) Define what is *relation R* and indicate exactly when a relation, is a *function*.
(b) Let F and G be functions. Then $G \circ F$ is automatically a relation. Prove that $G \circ F$ is also a function.
- (20) 5(a) Define what it means for the total function $f: A \rightarrow B$ to be *injective* and what it means for it to be *surjective*.
(b) Let $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{4\}$ be the function defined by $f(x) = (4x-7)/(x-3)$. Prove that f is a *bijective total function*.
- (15) 6(a) Let $\langle A_i : i \in I \rangle$ be an *indexed family* of sets. Define what are $(\bigcup_{i \in I} A_i)$ and $(\bigcap_{i \in I} A_i)$ by using bounded quantifiers.
(b) Prove that for any set B, $B - (\bigcap_{i \in I} A_i) = \bigcup_{i \in I} (B - A_i)$

Solutions to Test #1

Fall 2019

1(a) Let $A = \text{Adam is guilty}$, $B = \text{Ben is guilty}$, $C = \text{Chad is guilty}$.

The argument says $[(A \rightarrow (\neg C)) \wedge (A \vee B)] \Rightarrow (C \rightarrow B)$.

(b)	A	B	C	$[(A \rightarrow (\neg C)) \wedge (A \vee B)] \rightarrow (C \rightarrow B)$
0	0	0	0	1
0	0	1	0	0
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	0
1	1	0	0	1
1	1	1	0	1

Since $[(A \rightarrow (\neg C)) \wedge (A \vee B)] \rightarrow (C \rightarrow B)$ is a tautology, the argument is valid.

2(a) $(\exists !x)[Q(x)]$ means $(\exists x)[Q(x) \wedge (\forall y)\{Q(y) \rightarrow (x=y)\}]$.

$(\forall a \in A)[R(x)]$ means $(\forall x)[x \in A \rightarrow R(x)]$

$$(b) \neg(\forall x)(\exists y)[\{f(x) = f(y)\} \rightarrow \{\neg(x=y) \wedge (x+y=0)\}]$$

$$\Leftrightarrow (\exists x)\neg(\exists y)[\{f(x) = f(y)\} \rightarrow \{\neg(x=y) \wedge (x+y=0)\}] \text{ Quantifier Neg. law}$$

$$\Leftrightarrow (\exists x)(\forall y)\neg[\neg\{f(x) = f(y)\} \vee \{\neg(x=y) \wedge (x+y=0)\}] \text{ Quantifier Neg. \& Cond. law}$$

$$\Leftrightarrow (\exists x)(\forall y)[\neg\neg\{f(x) = f(y)\} \wedge \neg\{\neg(x=y) \wedge (x+y=0)\}] \text{ De Morgan's law}$$

$$\Leftrightarrow (\exists x)(\forall y)[\{f(x) = f(y)\} \wedge \{\neg\neg(x=y) \vee \neg(x+y=0)\}] \text{ De Morgan's law}$$

$$\Leftrightarrow (\exists x)(\forall y)[\{f(x) = f(y)\} \wedge \{(x=y) \vee \neg(x+y=0)\}] \text{ Double Neg. law}$$

$$3(a) R^{-1} = \{(b, a); (a, b) \in R\} \quad S \circ R = \{(a, c); (\exists b)[(a, b) \in R \wedge (b, c) \in S]\}.$$

(b) For each $a \in \mathbb{Z}$, we have $a^2 - a^2 = 0 = 10(0)$. $\therefore aRa$ for each $a \in \mathbb{Z}$. Hence R is reflexive.

Now suppose aRb . Then $a^2 - b^2 = 10k$ for some $k \in \mathbb{Z}$.

Then $b^2 - a^2 = -10k = 10(-k)$. $\therefore bRa$ bec. $(-k) \in \mathbb{Z}$.

Hence $(\forall a, b \in \mathbb{Z})[aRb \rightarrow bRa]$. $\therefore R$ is symmetric.

Finally suppose $aRb \wedge bRc$. Then $a^2 - b^2 = 10k$ and $b^2 - c^2 = 10l$ for some $k, l \in \mathbb{Z}$. Hence $a^2 - c^2 = 10(k+l)$, so aRc .

$\therefore (\forall a, b, c \in \mathbb{Z})(aRb \wedge bRc \rightarrow aRc)$. $\therefore R$ is transitive. Hence R is an equivalence relation on \mathbb{Z} .

3(c) Let $[a]_R = \{x \in \mathbb{Z} : xRa\}$ and $[a]_{10} = \{x \in \mathbb{Z} : x \equiv a \pmod{10}\}$. Then

$$0^2 \equiv 0 \pmod{10}, \quad 1^2 \equiv 1 \pmod{10}, \quad 2^2 \equiv 4 \pmod{10}, \quad 3^2 \equiv 9 \pmod{10}$$

$$4^2 = 16 \equiv 6 \pmod{10}, \quad 5^2 = 25 \equiv 5 \pmod{10}, \quad 6^2 = 36 \equiv 6 \pmod{10}$$

$$7^2 = 49 \equiv 9 \pmod{10}, \quad 8^2 = 64 \equiv 4 \pmod{10}, \quad 9^2 = 81 \equiv 1 \pmod{10}$$

$$\therefore [0]_R = [0]_{10} = \{10k : k \in \mathbb{Z}\}, \quad [1]_R = [1]_{10} \cup [9]_{10} = \{10k+1 : k \in \mathbb{Z}\}$$

$$[2]_R = [2]_{10} \cup [8]_{10} = \{10k+2 : k \in \mathbb{Z}\}, \quad [3]_R = [3]_{10} \cup [7]_{10} = \{10k+3 : k \in \mathbb{Z}\}$$

$$[4]_R = [4]_{10} \cup [6]_{10} = \{10k+4 : k \in \mathbb{Z}\}, \quad [5]_R = [5]_{10} = \{10k+5 : k \in \mathbb{Z}\}.$$

4(a) A relation R is a set of ordered pairs. A relation F is a function if $(\forall a)(\forall b_1)(\forall b_2) \{ (a, b_1) \in F \wedge (a, b_2) \in F \} \rightarrow (b_1 = b_2)$

(b) Suppose $(a, c_1) \in G \circ F$ and $(a, c_2) \in G \circ F$. Then we can find b_1 & b_2 such that $(a, b_1) \in F \wedge (b_1, c_1) \in G$ and $(a, b_2) \in F$ and $(b_2, c_2) \in G$. Since $(a, b_1) \in F \wedge (a, b_2) \in F$, it follows that $b_1 = b_2$ because F is a function. So $(b_1, c_1) \in G \wedge (b_1, c_2) \in G$. Hence $c_1 = c_2$ because G is a function. So for any a, c_1, c_2 we have $\{(a, c_1) \in G \circ F \wedge (a, c_2) \in G \circ F\} \rightarrow (c_1 = c_2)$. Hence $G \circ F$ is a function.

5(a) $f: A \rightarrow B$ is injective if $(\forall x_1, x_2 \in A) [x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)]$

This is equivalent to saying $(\forall x_1, x_2 \in A) [\{f(x_1) = f(x_2)\} \rightarrow (x_1 = x_2)]$.

$f: A \rightarrow B$ is surjective if $(\forall y \in B)(\exists x \in A) [f(x) = y]$.

(b) Suppose $f(x_1) = f(x_2)$. Then $(4x_1 - 7)/(x_1 - 3) = (4x_2 - 7)/(x_2 - 3)$

$$\text{So } 4x_1x_2 - 12x_1 - 7x_2 + 21 = 4x_1x_2 - 7x_1 - 12x_2 + 21$$

$$\therefore -5x_1 = -5x_2. \text{ So } x_1 = x_2. \text{ Hence } f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{4\} \text{ is injective.}$$

Let $y \in \mathbb{R} - \{4\}$. Put $x = 3 + 5/(y-4)$. Then x is well-defined and $x \in \mathbb{R} - \{3\}$ because $5/(y-4)$ is never zero. [We get this x by solving the equation $y = f(x) = (4x-7)/(x-3)$ for x in terms of y . This is done aside to make the argument clear.]

$$\begin{aligned}
 5(b) \text{ So } f(x) = \frac{4x-7}{x-3} &= \frac{4\left(3 + \frac{5}{y-4}\right) - 7}{\left(3 + \frac{5}{y-4}\right) - 3} = \frac{5 + 20/(y-4)}{5/(y-4)} \\
 &= \frac{5(y-4) + 20}{y-4} \cdot \frac{y-4}{5} = \frac{5y}{5} = y.
 \end{aligned}$$

Hence $(\forall y \in R - \{4\}) (\exists x \in R - \{3\}) [f(x) = y]$. Hence $f: R - \{3\} \rightarrow R - \{4\}$ is surjective. Thus $f: R - \{3\} \rightarrow R - \{4\}$ is bijective.

Aside: $y = (4x-7)/(x-3) \therefore (x-3)y = 4x-7$.

$$\begin{aligned}
 \therefore xy - 4x &= 3y - 7 \quad \therefore x(y-4) = 3y-7 \quad \therefore x = \frac{3y-7}{y-4} \\
 \therefore x &= \frac{3y-12+5}{y-4} = \frac{3(y-4)+5}{y-4} = 3 + \frac{5}{y-4}.
 \end{aligned}$$

$$6(a) \bigcup_{i \in I} A_i = \{x : (\exists i \in I)(x \in A_i)\}, \bigcap_{i \in I} A_i = \{x : (\forall i \in I)(x \in A_i)\}$$

6(b) Let $x \in B - \left(\bigcap_{i \in I} A_i\right)$. Then $x \in B$ and $x \notin \bigcap_{i \in I} A_i$. So

$x \in B$ and $x \notin A_{i_0}$ for at least one $i_0 \in I$. $\therefore x \in (B - A_{i_0})$

$\therefore x \in \bigcup_{i \in I} (B - A_i)$. Hence $B - \left(\bigcap_{i \in I} A_i\right) \subseteq \bigcup_{i \in I} (B - A_i)$ (*)

Now let $x \in \bigcup_{i \in I} (B - A_i)$. Then $x \in (B - A_{i_0})$ for at least one $i_0 \in I$.

So $x \in B$ and $x \notin A_{i_0}$. $\therefore x \in B$ and $x \notin \bigcap_{i \in I} A_i$.

$\therefore x \in B - \left(\bigcap_{i \in I} A_i\right)$. Hence $\bigcup_{i \in I} (B - A_i) \subseteq B - \left(\bigcap_{i \in I} A_i\right)$... (**)

From (*) & (**) it follows that $B - \left(\bigcap_{i \in I} A_i\right) = \bigcup_{i \in I} (B - A_i)$.

6(b) [another proof of 6(b)]

$$\begin{aligned}
 x \in B - \left(\bigcap_{i \in I} A_i\right) &\iff x \in B \wedge \neg [\bigcap_{i \in I} (x \in A_i)] \\
 &\iff x \in B \wedge \neg [(\forall i \in I)(x \in A_i)] \\
 &\iff x \in B \wedge (\exists i \in I)[\neg (x \in A_i)] \\
 &\iff (\exists i \in I)[x \in B \wedge \neg (x \in A_i)] \\
 &\iff (\exists i \in I)[x \in (B - A_i)] \\
 &\iff x \in \bigcup_{i \in I} (B - A_i)
 \end{aligned}$$

$$\therefore B - \left(\bigcap_{i \in I} A_i\right) = \bigcup_{i \in I} (B - A_i)$$

END.