

Answer all 6 questions. No Calculators or Cell phones are allowed. An unjustified answer will receive little or no credit. BEGIN EACH OF THE 6 QUESTIONS ON 6 SEPARATE PAGES.

- (15) 1. (a) Define what is a *finite sequence f* and define what is a *subsequence of f*.
(b) Define what it means for $f: A \rightarrow B$ to be *surjective*. Prove that if $f: A \rightarrow B$ and $g: B \rightarrow C$ are both surjective, then $g \circ f: A \rightarrow C$ is also surjective.
- (20) 2. (a) Let $f: X \rightarrow Y$ be a function and suppose that $A, B \subseteq X$ and $C, D \subseteq Y$.
Define what is $f[A]$ and what is $f^{-1}[C]$.
(b) Is it always true that $f[A \cap B] \subseteq f[A] \cap f[B]$?
(c) Is it always true that $f^{-1}[C - D] \subseteq f^{-1}[C] - f^{-1}[D]$?
- (15) 3. (a) Write down what is the *First Principle of Mathematical Induction* for \mathbb{N} .
(b) Prove that $(\forall n \in \mathbb{N}) (7^n - 6n - 1)$ is an integer multiple of 36.
- (15) 4. (a) Define what it means for A to be a *finite set* & for B to be a *denumerable set*.
(b) Prove that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is a denumerable set. [If you claim that a function is a *bijection*, you must prove that this is indeed so.]
- (20) 5(a) Define what it means for the infinite sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ to be *convergent*.
(b) Suppose $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to A and $\langle b_n \rangle_{n \in \mathbb{N}}$ converges to B . Prove that $\langle a_n + b_n \rangle_{n \in \mathbb{N}}$ converges to $A+B$.
(You may use the fact that a convergent sequence is bounded.)
- (15) 6(a) Define what it means for the sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ to be a *Cauchy sequence*.
(b) Prove that if $\langle a_n \rangle_{n \in \mathbb{N}}$ is a convergent sequence, then it is a Cauchy sequence.

$\in \forall \exists \Delta \oplus \subseteq \notin \subset \rightarrow \neg \neq \infty \emptyset \equiv \approx \leftrightarrow \times \aleph \sqrt{} \nabla \square \cong \perp \pm \geq \leq \circ \uparrow \downarrow \perp \dashv \cup \cap \mathbb{R} \mathbb{Z} \langle \rangle \mathbb{N}$

Solutions to Test #2

Fall 2019.

1(a) A finite sequence is any function $f: \mathbb{N}_n \rightarrow A$ where A is any set & $n \in \mathbb{N}$. It is usually written as $\langle f(0), f(1), \dots, f(n-1) \rangle$. A subsequence of f is an ordered pair (f, g) where $g: \mathbb{N}_k \rightarrow \mathbb{N}_n$ is a strictly increasing function and $\mathbb{N}_n = \text{dom}(f)$. It is usually written as $\langle f(g(0)), f(g(1)), \dots, f(g(k-1)) \rangle$.

(b) $f: A \rightarrow B$ is surjective if $(\forall b \in B)(\exists a \in A)[b = f(a)]$. Thus $g \circ f: A \rightarrow C$ will be surjective if $(\forall c \in C)(\exists a \in A)[c = f(a)]$. So let $c \in C$. Since $g: B \rightarrow C$ is surjective, we can find a $b \in B$ such that $c = g(b)$. Also since $f: A \rightarrow B$ is surjective and $b \in B$, we can find an $a \in A$ such that $b = f(a)$. Hence $c = g(b) = g(f(a)) = (g \circ f)(a)$. So $g \circ f: A \rightarrow C$ is surjective.

2(a) $f[A] = \{y \in Y : (\exists x \in A)[y = f(x)]\}, \quad f^{-1}[C] = \{x \in X : f(x) \in C\}$.

(b) YES. Let $y \in f[A \cap B]$. Then we can find an $x \in A \cap B$ such that $y = f(x)$. Since $x \in A \cap B$, $x \in A$ and $x \in B$. So $y \in f[A]$ because $y = f(x) \& x \in A$. And $y \in f[B]$ because $y = f(x) \& x \in B$. $\therefore y \in f[A] \cap f[B]$.

i. $y \in f[A \cap B] \Rightarrow y \in f[A] \cap f[B]$. Hence $f[A \cap B] \subseteq f[A] \cap f[B]$.

(c) YES. Let $x \in f^{-1}[C \cap D]$. Then $f(x) \in C \cap D$. So $f(x) \in C$ and $f(x) \in D$. Thus $x \in f^{-1}[C]$ and $x \notin f^{-1}[D]$ (bec. if $x \in f^{-1}[D]$ we would have had $f(x) \in D$). $\therefore x \in f^{-1}[C] - f^{-1}[D]$. Hence $x \in f^{-1}[C \cap D] \Rightarrow x \in f^{-1}[C] - f^{-1}[D]$. $\therefore f^{-1}[C \cap D] \subseteq f^{-1}[C] - f^{-1}[D]$.

3(a) Let $P(n)$ be a formula of first order logic with free variable n . Then $\{P(0) \wedge (\forall n \in \mathbb{N})[P(n) \rightarrow P(n+1)]\} \Rightarrow (\forall n \in \mathbb{N})[P(n)]$.

(b) Let $P(n)$ be the formula $(\exists k \in \mathbb{N})[7^n - 6n - 1 = 36k]$. Then $7^0 - 6(0) - 1 = 1 - 0 - 1 = 0 = 36(0)$. So $P(0)$ is true.

Now suppose $P(n)$ is true. Then $7^n - 6n - 1 = 36k$ for some $k \in \mathbb{N}$.

$$3(b) \text{ So } 7^{n+1} - 6(n+1) - 1 = 7 \cdot 7^n - 6n - 6 - 1 = 7 \cdot 7^n - 7 \cdot 6n - 7 + 6 \cdot 6n \\ = 7(7^n - 6n - 1) + 36n = 7(36k) + 36n = 36(7k+n),$$

Hence if $P(n)$ is true, then $P(n+1)$ will be true. So by the First Principle of Math Induction, $(\forall n \in \mathbb{N})[P(n)] \therefore (\forall n \in \mathbb{N})[7^n - 6n - 1 \text{ is a mult. of 36}]$

4(a) A is finite if we can find a bijection $f: A \rightarrow \mathbb{N}_n$ for some $n \in \mathbb{N}$.
 B is denumerable if we can find a bijection $g: B \rightarrow \mathbb{N}$.

(b) Let $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{N}$ be defined by $f(k, l) = 2^{k-1}(2l-1) - 1$.

Now suppose $f(k_1, l_1) = f(k_2, l_2)$. Then $2^{k_1-1}(2l_1-1) - 1 = 2^{k_2-1}(2l_2-1) - 1$.

$\therefore 2^{k_1-1}(2l_1-1) = 2^{k_2-1}(2l_2-1)$. Since $(2l_1-1)$ & $(2l_2-1)$ are both odd, we must have $2^{k_1-1} = 2^{k_2-1}$. So $k_1-1 = k_2-1 \therefore k_1 = k_2$.

$\therefore 2l_1-1 = 2l_2-1 \Rightarrow 2l_1 = 2l_2 \Rightarrow l_1 = l_2 \therefore (k_1, l_1) = (k_2, l_2)$.

So f is injective. Now let n be any element of \mathbb{N} . Then we can write $n+1$ in the form $2^a(2b+1)$ with $a, b \in \mathbb{N}$ bec. any positive integer is a power of 2 times an odd number.

$\therefore n+1 = 2^a \cdot (2b+1) \Rightarrow n = 2^{(a+1)-1} \cdot [2(b+1)-1] - 1$. So

$n = f(a+1, b+1)$ where $a+1$ & $b+1$ are in \mathbb{Z}^+ . Hence f is a surjective function. $\therefore f: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{N}$ is a bijection. So $\mathbb{Z}^+ \times \mathbb{Z}^+$ is denumerable.

5(a) $\langle a_n \rangle_{n \in \mathbb{N}}$ is convergent if $(\exists L \in \mathbb{R})(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|a_n - L| < \epsilon)$.

(b) Let $\epsilon > 0$ be given. We want to find an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n \cdot b_n - A \cdot B| < \epsilon$. Now since $\langle a_n \rangle$ is convergent, then $\langle a_n \rangle$ is bounded. So we can find a real number $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$. Let $\epsilon' = \epsilon / (M + |B|)$.

Since $M > 0$ and $|B| \geq 0$, $\epsilon' > 0$. Now since $\langle a_n \rangle$ converges to A , we can find an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - A| < \epsilon'$. Also since $\langle b_n \rangle$ converges to $|B|$, we can find an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - B| < \epsilon'$.

5(b) Let $N = \max\{N_1, N_2\}$. Then $N \in \mathbb{N}$ and for all $n \geq N$,

$$\begin{aligned} |a_n \cdot b_n - A \cdot B| &= |a_n \cdot b_n - a_n \cdot B + a_n \cdot B - A \cdot B| \\ &= |a_n(b_n - B) + B(a_n - A)| \\ &\leq |a_n| \cdot |b_n - B| + |B| \cdot |a_n - A| \\ &\leq M \cdot |b_n - B| + |B| \cdot |a_n - A| \\ &< M \cdot \varepsilon' + |B| \cdot \varepsilon' = (M + |B|) \cdot \varepsilon' = \varepsilon. \end{aligned}$$

So $(\forall n \geq N)(|a_n \cdot b_n - A \cdot B| < \varepsilon)$. Hence $\langle a_n \cdot b_n \rangle$ converges to $A \cdot B$.

6(a) $\langle a_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \geq N)[|a_m - a_n| < \varepsilon].$$

(b) Let $\varepsilon > 0$ be given. We want to find an $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_m - a_n| < \varepsilon$. Now since $\varepsilon > 0$, it follows that $\varepsilon/2 > 0$. Since $\langle a_n \rangle$ is convergent, we can find a number L to which $\langle a_n \rangle$ converges.

Also since $\varepsilon/2 > 0$, we can find an $N \in \mathbb{N}$ such that for all $k \geq N$, $|a_k - L| < \varepsilon/2$. So for all $m, n \geq N$,

$$\begin{aligned} |a_m - a_n| &= |(a_m - L) + (L - a_n)| \\ &\leq |a_m - L| + |L - a_n| \\ &= |a_m - L| + |a_n - L| \\ &< \varepsilon/2 + \varepsilon/2 \quad \text{because } m \geq N \text{ & } n \geq N \\ &= \varepsilon. \end{aligned}$$

∴ for all $m, n \geq N$; $|a_m - a_n| < \varepsilon$. Hence $\langle a_n \rangle$ is a Cauchy sequence. END.