

Answer all 6 questions. No calculators, notes, or on-line are allowed. An unjustified answer will receive little or no credit. Draw a line to separate each of your 6 solutions to the 6 questions.

- (15) 1(a) Translate the following argument into ***symbolic language***.
"If Beth migrates, then Cindy will not migrate. Either Amy or Beth will migrate. Therefore, if Cindy migrates, then Amy will migrate."
- (b) Use a ***truth table*** to determine if this argument is ***logically valid***.
- (15) 2(a) Define $(\exists x \in A)[R(x)]$ and $(\forall x \in A)[S(x)]$ in terms of *unbounded quantifiers*.
(b) Convert the formula $\neg(\forall y)(\exists z)[\{h(y) < h(z)\} \rightarrow \{(y+z=4) \wedge \neg(y=z)\}]$ into a *logically equivalent formula* in which no " \neg " sign **governs** a quantifier or a connective. [Specify which law you use at each step.]
- (15) 3(a) Let $\langle B_i : i \in I \rangle$ be an *indexed family* of subsets of a universal set U. Define what are $(\bigcup_{i \in I} B_i)$ and $(\bigcap_{i \in I} B_i)$. Also define what is $(B_i)^c$.
(b) Prove that $[\bigcap_{i \in I} B_i]^c = \bigcup_{i \in I} [(B_i)^c]$.
- (15) 4(a) If R & S are relations define what are S^{-1} & $R \circ S$. Define what is a *function F*.
(b) Let F and G be any functions. Prove that $(F \circ G)$ is also a function.
- (20) 5(a) Define what is an *equivalence relation R* on a set A.
(b) Let R be the relation on \mathbb{Z} defined by aRb if $(a^4 - b^4)$ is an integer multiple of 6. Prove that R is an *equivalence relation* and find the *equivalence classes* into which R partitions \mathbb{Z} . (Specify each equivalence class, completely.)
- (20) 6(a) Define what it means for the *partial function* $f: A \rightarrow B$ to be a *total function*. When exactly is f *injective* and when exactly is f *surjective*?
(b) Let $f: \mathbb{R} - \{5\} \rightarrow \mathbb{R} - \{3\}$ be the partial function defined by $f(x) = (3x-4)/(x-5)$. Prove that $f: \mathbb{R} - \{5\} \rightarrow \mathbb{R} - \{3\}$ is a *total, injective, and surjective* function.

END

1(a) Let $A = \text{Amy migrates}$, $B = \text{Beth migrates}$, and $C = \text{Cindy migrates}$

The argument says: $[\{B \rightarrow (\neg C)\} \wedge (A \vee B)] \Rightarrow (C \rightarrow A)$. corresp. proposition

$$(b) \begin{array}{|c|c|c|} \hline A & B & C \\ \hline \end{array} \quad [\{B \rightarrow (\neg C)\} \wedge (A \vee B)] \rightarrow (C \rightarrow A) \leftarrow$$

1	1	1	0	0	0	1	1	1
1	1	0	1	1	1	1	1	1
1	0	1	1	0	1	1	1	1
1	0	0	1	1	1	1	1	1
0	1	1	0	0	0	1	1	0
0	1	0	1	1	1	1	1	1
0	0	1	1	0	0	1	1	1
0	0	0	1	1	0	1	1	1

Since the corresponding proposition is a tautolog, the argument is valid.

$$2(a) (\exists x \in A)[R(x)] \text{ means } (\exists x)[(x \in A) \wedge R(x)]$$

$$(\forall x \in A)[S(x)] \text{ means } (\forall x)[(x \in A) \rightarrow S(x)]$$

$$(b) \neg (\forall y)(\exists z) [\{h(y) < h(z)\} \rightarrow \{(y+z=4) \wedge \neg(y=z)\}]$$

$$\Leftrightarrow (\exists y)\neg(\exists z) [\neg\{h(y) < h(z)\} \vee \{(y+z=4) \wedge \neg(y=z)\}]$$

\forall -quantifier negation law & conditional law

$$\Leftrightarrow (\exists y)(\forall z) \neg [\neg\{h(y) < h(z)\} \vee \{(y+z=4) \wedge \neg(y=z)\}]$$

\exists -quantifier negation law

$$\Leftrightarrow (\exists y)(\forall z) [\neg\neg\{h(y) < h(z)\} \wedge \neg\{(y+z=4) \wedge \neg(y=z)\}]$$

De Morgan's law $\neg(A \vee B)$ & double negation law

$$\Leftrightarrow (\exists y)(\forall z) [\{h(y) < h(z)\} \wedge \{\neg(y+z=4) \vee \neg\neg(y=z)\}]$$

$$\Leftrightarrow (\exists y)(\forall z) [\{h(y) < h(z)\} \wedge \{\neg(y+z=4) \vee (y=z)\}]$$

$$3(a) \bigcup_{i \in I} B_i = \{x : (\exists i \in I)(x \in B_i)\}, \bigcap_{i \in I} B_i = \{x : (\forall i \in I)(x \in B_i)\}, B_i^c = \{x \in U : x \notin B_i\}.$$

$$(b) x \in \left[\bigcap_{i \in I} B_i \right]^c \Leftrightarrow (x \in U) \wedge (x \notin \bigcap_{i \in I} B_i) \Leftrightarrow (x \in U) \wedge \{\neg(\forall i \in I)(x \in B_i)\}$$

$$\Leftrightarrow (x \in U) \wedge (\exists i \in I) \{\neg(x \in B_i)\}$$

$$\Leftrightarrow (\exists i \in I) [x \in U \wedge x \notin B_i]$$

$$\Leftrightarrow (\exists i \in I) (x \in B_i^c) \Leftrightarrow x \in \bigcup_{i \in I} (B_i^c)$$

$$\therefore \left(\bigcap_{i \in I} B_i \right)^c = \bigcup_{i \in I} (B_i^c).$$

$$4(a) S^{-1} = \{(b, a) : (a, b) \in S\}, R \circ S = \{(a, c) : (\exists b)[(a, b) \in S \wedge (b, c) \in R]\}$$

A function is a relation F such that $[(a, c_1) \in F \wedge (a, c_2) \in F] \Rightarrow (c_1 = c_2)$.

(b) Suppose $(a, c_1) \in F \circ G$ and $(a, c_2) \in F \circ G$. Then we can find a b_1 such that $(a, b_1) \in G$ and $(b_1, c_1) \in F$; and we can find a b_2 such that $(a, b_2) \in G$ and $(b_2, c_2) \in F$. Since G is a function and $(a, b_1) \in G \wedge (a, b_2) \in G$ we must have $b_1 = b_2$. Also since $b_1 = b_2$ & F is a function and $(b_1, c_1) \in F \wedge (b_1, c_2) \in F$ we must have $c_1 = c_2$. So $(a, c_1) \in F \circ G \wedge (a, c_2) \in F \circ G \Rightarrow c_1 = c_2$. Hence $F \circ G$ is also a function.

5(a) An relation R on A is an equivalence relation if

(i) R is reflexive on A , i.e., $(\forall a \in A)[aRa]$

(ii) R is symmetric on A , i.e., $(\forall a, b \in A)[aRb \rightarrow bRa]$

& (iii) R is transitive on A , i.e., $(\forall a, b, c \in A)[aRb \wedge bRc \rightarrow aRc]$.

(b) Since $a^4 - a^4 = 0 = 6(0)$, we have aRa for all $a \in \mathbb{Z}$.

Suppose aRb . Then $a^4 - b^4 = 6k$ for some $k \in \mathbb{Z}$. So

$$b^4 - a^4 = -(a^4 - b^4) = -6k = 6(-k). \text{ Since } -k \in \mathbb{Z}, \text{ it follows that } bRa$$

Finally suppose aRb & bRc . Then $a^4 - b^4 = 6k$ and $b^4 - c^4 = 6l$ for some $k, l \in \mathbb{Z}$. So $a^4 - c^4 = (a^4 - b^4) + (b^4 - c^4) = 6k + 6l = 6(k+l)$. Since $k+l \in \mathbb{Z}$, it follows that aRc .

Hence R is an equivalence relation on \mathbb{Z} .

We have $0^4 \equiv 0 \pmod{6}$, $1^4 \equiv 1 \pmod{6}$, $2^4 \equiv 16 \equiv 4 \pmod{6}$

$3^4 \equiv (9)(9) \equiv (3)(3) \equiv 3 \pmod{6}$, $4^4 \equiv (16)(16) \equiv (4)(4) \equiv 4 \pmod{6}$, and

$5^4 \equiv (25)(25) \equiv (1)(1) \equiv 1 \pmod{6}$. So the equivalence classes are:

$$[0]_R = [0]_6 = \{6k : k \in \mathbb{Z}\}, [3]_R = [3]_6 = \{6k+3 : k \in \mathbb{Z}\}$$

$$[1]_R = [1]_6 \cup [5]_6 = \{6k+1 : k \in \mathbb{Z}\} \cup \{6k+5 : k \in \mathbb{Z}\} = \{6k+1 : k \in \mathbb{Z}\}$$

$$[2] = [2]_6 \cup [4]_6 = \{6k+2 : k \in \mathbb{Z}\} \cup \{6k+4 : k \in \mathbb{Z}\} = \{6k+2 : k \in \mathbb{Z}\}$$

6(a) $f: A \rightarrow B$ is a total function if $(\forall a \in A)(\exists b \in B)[(a, b) \in f]$.

$f: A \rightarrow B$ is injective if $(\forall a_1, a_2 \in A)[\{f(a_1) = f(a_2)\} \rightarrow (a_1 = a_2)]$

$f: A \rightarrow B$ is surjective if $(\forall b \in B)(\exists a \in A)[f(a) = b]$

(b) $f: R - \{5\} \rightarrow R - \{3\}$ with $f(x) = (3x-4)/(x-5) = \frac{3x-15+11}{x-5} = \frac{3+11}{x-5}$

f is total because for all $x \in R - \{5\}$, $f(x)$ is defined
and $f(x) \in R - \{3\}$ because $\frac{11}{x-5}$ can never be 3.

Suppose $f(x_1) = f(x_2)$. Then $3 + \frac{11}{x_1-5} = 3 + \frac{11}{x_2-5}$.

So $\frac{11}{x_1-5} = \frac{11}{x_2-5}$ and thus $\frac{x_1-5}{x_2-5} = \frac{11}{11}$.

Hence $x_1-5 = x_2-5$ and so $x_1 = x_2$ since $f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$, it follows that $f: R - \{5\} \rightarrow R - \{3\}$ is injective.

Let $y \in R - \{3\}$. We will find an $x \in R - \{5\}$ such that $f(x) = y$.

Suppose $y = f(x) = 3 + \frac{11}{x-5}$. Then $y-3 = \frac{11}{x-5}$

So $\frac{y-3}{11} = \frac{1}{x-5}$. Thus $x-5 = \frac{11}{y-3}$ & so $x = 5 + \frac{11}{y-3} \in R - \{5\}$

$$\begin{aligned} \text{Now } f(x) &= \frac{3\left(5 + \frac{11}{y-3}\right) - 4}{5 + \frac{11}{y-3} - 5} = \frac{15 - 4 + 33/(y-3)}{11/(y-3)} \\ &= \frac{11(y-3) + 33}{11} = \frac{11y}{11} = y. \end{aligned}$$

So for any $y \in R - \{3\}$, we found an $x = 5 + \frac{11}{y-3} \in R - \{5\}$

such that $f(x) = y$. (Note: $x \in R - \{5\}$ because $\frac{11}{y-3} \neq 0$)

Hence $f: R - \{5\} \rightarrow R - \{3\}$ is surjective.

END