MAA 3200 - INTROD TO ADVANCED MATH FLORIDA INT'L UNIV.
TEST #1 - SPRING 2021 TIME: 75 min.

Answer all 6 questions. **No calculators, notes, or on-line are allowed.** *An unjustified answer will receive little or no credit.* ***Draw a line to separate each of your 6 solutions to the 6 questions.*** (Double check the solutions that you send as a pdf file – to ensure it contains everything.)

(15) 1(a) Translate the following argument into *symbolic language*.
"Either Amy or Beth will marry. If Amy marries, then Celia will not marry. Therefore, if Celia marries, then Beth will marry."
(b) Use a ***truth table*** to determine if this argument is *logically valid*.

(15) 2(a) Define $(\exists x \in B)[Q(x)]$ and $(\forall x \in B)[R(x)]$ in terms of *unbounded quantifiers*.
(b) Convert the formula $\neg(\exists y)(\forall z)\,[\{g(y) > g(z)\} \to \{(y.z = 0 \;\wedge\; \neg(y = z)\}]$ into a *logically equivalent formula* in which no "$\neg$" sign **governs** a *quantifier* or a *connective*. [*Specify which law you use at each step*.]

(15) 3(a) Define what is a *relation*. *If R & S* are relations define what are S^{-1} and $R \circ S$.
(b) Let R and S be any relations. Prove that $(R \circ S)^{-1} = (S^{-1}) \circ (R^{-1})$.

(20) 4(a) Explain what's the difference between a ***function*** and a *function on the* ***set A***.
(b) Let R be the relation on $\mathbb{Z}$ defined by aRb *if* $(a^3 - b^3)$ is an *integer multiple* of 9. Prove that R is an *equivalence relation* and find the *equivalence classes* into which R partitions $\mathbb{Z}$. (*Specify each equivalence class, completely.*)

(20) 5(a) Define what it means for the *partial* function $f: A \to B$ to be a *total function.* When exactly is f *injective* and when exactly is f *surjective* ?
(b) Let $f: \mathbb{R}-\{3\} \to \mathbb{R}-\{2\}$ be the partial function defined by $f(x) = (2x-5)/(x-3)$. Prove that $f: \mathbb{R}-\{3\} \to \mathbb{R}-\{2\}$ is a *total*, *injective*, and *surjective* function.

(15) 6(a) Let $\langle B_i : i \in I\rangle$ be an *indexed family* of sets. Define what are $(\bigcup_{i\in I} B_i)$ and $(\bigcap_{i\in I} B_i)$ by using bounded quantifiers.
(b) Prove that for any set A, $\bigcap_{i\in I}(A - B_i) = A - (\bigcup_{i\in I} B_i)$. END

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1 (a) Let A = Amy ~~marries~~ marries, B = Beth marries, and C = Celia marries.
The argument says: $[(A \vee B) \wedge (A \to \neg C)] \Rightarrow (C \to B)$

(b)

A	B	C	$[(A \vee B)$	$\wedge$	$(A \to$	$\neg C)]$	$\to$	$(C \to B)$
1	1	1	1	0	0	0	1	1
1	1	0	1	1	1	1	1	1
1	0	1	1	0	0	0	1	0
1	0	0	1	1	1	1	1	1
0	1	1	1	1	1	0	1	1
0	1	0	1	1	1	1	1	1
0	0	1	0	0	1	0	1	0
0	0	0	0	0	1	1	1	1

So the argument is logically valid. ↖ tautology.

2 (a) $(\exists x \in B)\, Q(x)$ means $(\exists x)(x \in B \wedge Q(x))$.
$(\forall x \in B)\, R(x)$ means $(\forall x)(x \in B \to R(x))$.

(b) $\neg (\exists y)(\forall z)\, [\{g(y) > g(z)\} \to \{(y.z = 0) \wedge \neg(y = z)\}]$

$\Leftrightarrow (\forall y) \neg (\forall z)\, [\{g(y) > g(z)\} \to \{(y.z = 0) \wedge \neg(y = z)\}]$
Quantifier Negation law

$\Leftrightarrow (\forall y)(\exists z) \neg [\neg \{g(y) > g(z)\} \vee \{(y.z) = 0 \wedge \neg(y = z)\}]$
Quantifier Neg. law & Conditional law

$\Leftrightarrow (\forall y)(\exists z)\, [\neg\neg\{g(y) > g(z)\} \wedge \{\neg(y.z = 0) \vee \neg\neg(y = z)\}]$
De Morgan's law

$\Leftrightarrow (\forall y)(\exists z)\, [\{g(y) > g(z)\} \wedge \{\neg(y.z = 0) \vee (y = z)\}]$
Double Negation law.

3 (a) A relation is just a set of ordered pairs
$S^{-1} = \{(b, a) : (a, b) \in S\}$ $R \circ S = \{(a, c) : (\exists b)[(a, b) \in S \wedge (b, c) \in R]\}$.

(b) $(c, a) \in (R \circ S)^{-1} \Leftrightarrow (a, c) \in (R \circ S)$
$\Leftrightarrow (\exists b)\, [(a, b) \in S \wedge (b, c) \in R]$
$\Leftrightarrow (\exists b)\, [(b, c) \in R \wedge (a, b) \in S]$
$\Leftrightarrow (\exists b)\, [(c, b) \in R^{-1} \wedge (b, a) \in S^{-1}]$
$\Leftrightarrow (c, a) \in (S^{-1}) \circ (R^{-1})$

$\therefore (R \circ S)^{-1} = (S^{-1}) \circ (R^{-1})$.

4(a) A *function* is a set of ordered pairs, F, such that $[(a,b)\in F \wedge (a,c)\in F] \Rightarrow (b=c)$. A *function f on the set A* is a function, f, with $dom(f) = A$ & $ran(f) \subseteq A$. Here $dom(f) = \{a : (a,b)\in f\}$ & $ran(f) = \{b : (a,b)\in f\}$.

(b) For each $a\in\mathbb{Z}$, $a^3 - a^3 = 0 = 9(0)$. $\therefore (\forall a\in\mathbb{Z})[aRa]$. Suppose aRb. Then $a^3 - b^3 = 9(k)$ for some $k\in\mathbb{Z}$. So $b^3 - a^3 = -(a^3-b^3) = -9(k) = 9(-k)$. Since $(-k)\in\mathbb{Z}$, bRa. $\therefore (\forall a,b\in\mathbb{Z})[aRb \rightarrow bRa]$. Finally suppose aRb & bRc. Then $\exists k,l\in\mathbb{Z}$ such that $a^3-b^3 = 9k$ and $b^3 - c^3 = 9l$. So $a^3 - c^3 = (a^3-b^3)+(b^3-c^3) = 9k+9l = 9(k+l)$. Since $k+l\in\mathbb{Z}$, aRc. $\therefore (\forall a,b,c\in\mathbb{Z})[(aRb \wedge bRc) \rightarrow aRc]$. Hence R is an equivalence relation on $\mathbb{Z}$.

(c) Let $a \equiv_9 b$ be the modulo "9" relation & $[x]_9$ be the equivalence class modulo 9 that contains x. Then

$0^3 \equiv_9 0$ $\quad 3^3 \equiv_9 9(3) \equiv_9 0$ $\quad 6^3 \equiv_9 36(6) \equiv_9 0$

$1^3 \equiv_9 1$ $\quad 4^3 \equiv_9 16(4) \equiv_9 28 \equiv_9 1$ $\quad 7^3 \equiv_9 49(7) \equiv_9 4(7) \equiv_9 1$

$2^3 \equiv_9 8$ $\quad 5^3 \equiv_9 25(5) \equiv_9 7(5) \equiv_9 8$ $\quad 8^3 \equiv_9 64(8) \equiv_9 1(8) \equiv_9 8$

So the equivalence classes are:

$[0]_R = [0]_9 \cup [3]_9 \cup [6]_9 = \{3k : k\in\mathbb{Z}\}$,

$[1]_R = [1]_9 \cup [4]_9 \cup [7]_9 = \{3k+1 : k\in\mathbb{Z}\}$, and

$[2]_R = [2]_9 \cup [5]_9 \cup [8]_9 = \{3k+2 : k\in\mathbb{Z}\}$.

5. (a) The partial function $f : A \rightarrow B$ is a *total function* if $(\forall a\in A)(\exists b\in B)[(a,b)\in f]$. f is *injective* if $(\forall a_1, a_2 \in A)[\{f(a_1) = f(a_2)\} \rightarrow (a_1 = a_2)]$. f is *surjective* if $(\forall b\in B)(\exists a\in A)[f(a) = b]$.

(b). To show that f is a total function we must show that $f(x)$ is well-defined for each $x\in\mathbb{R}-\{3\}$ and that $f(x)$ is always a member of $\mathbb{R}-\{2\}$. Now clearly $f(x) = (2x-5)/(x-3)$ is well-defined because $x\neq 3$.

5.(b) Also $\frac{2X-5}{X-3} = \frac{2(X-3)+1}{(X-3)} = 2 + \frac{1}{X-3}$. Since $\frac{1}{X-3}$ can never be 0, $f(x) = (2x-5)/(x-3)$ can never be 2. (p.3)
So f is a total function.

Now we will prove that f is injective. Suppose $f(X_1) = f(X_2)$. Then $\frac{2X_1-5}{X_1-3} = \frac{2X_2-5}{X_2-3}$. So
$(2X_1-5)(X_2-3) = (2X_2-5)(X_1-3)$.

$\therefore\ 2X_1X_2 - 6X_1 - 5X_2 + 15 = 2X_1X_2 - 6X_2 - 5X_1 + 15$.

$\therefore\ -5X_2 + 6X_2 = -5X_1 + 6X_1 \Rightarrow X_2 = X_1 \Rightarrow X_1 = X_2$

$\therefore$ f is injective.

Finally we will show that f is surjective. Let y be any element of $\mathbb{R} - \{2\}$. Take $x = 3 + 1/(y-2)$
(We get this x by solving $y = (2x-5)/(x-3)$ for x in terms of y)

Then $f(x) = \frac{2.[3 + 1/(y-2)] - 5}{[3 + 1/(y-2)] - 3} = \frac{1 + 2/(y-2)}{1/(y-2)} = \frac{(y-2)+2}{1} = y$.

$\therefore$ f is surjective.

6 (a) $\bigcup_{i\in I} B_i = \{x : (\exists i \in I)(x \in B_i)\}$, $\bigcap_{i\in I} B_i = \{x : (\forall i \in I)(x \in B_i)\}$.

(b)
$$
\begin{aligned}
x \in \left[A - \left(\bigcup_{i\in I} B_i\right)\right] &\Leftrightarrow (x\in A) \wedge \left(x \notin \bigcup_{i\in I} B_i\right) \\
&\Leftrightarrow (x\in A) \wedge \neg\left(x \in \bigcup_{i\in I} B_i\right) \\
&\Leftrightarrow (x\in A) \wedge \neg[(\exists i\in I)(x\in B_i)] \\
&\Leftrightarrow (x\in A) \wedge (\forall i\in I)[\neg(x\in B_i)] \\
&\Leftrightarrow (\forall i\in I)[(x\in A) \wedge \neg(x\in B_i)] \\
&\Leftrightarrow (\forall i\in I)[x\in A \wedge (x\notin B_i)] \\
&\Leftrightarrow (\forall i\in I)[x\in(A-B_i)] \Leftrightarrow x \in \bigcap_{i\in I}(A-B_i)
\end{aligned}
$$

$\therefore\ \bigcap_{i\in I}(A-B_i) = A - \left(\bigcup_{i\in I} B_i\right)$.

END.

[5(b) $y = (2x-5)/(x-3) \Rightarrow y(x-3) = 2x-5 \Rightarrow yx - 2x = 3y-5$
$\Rightarrow x(y-2) = 3y-5 \Rightarrow x = \frac{3y-5}{y-2} = \frac{3(y-2)+1}{y-2} = 3 + \frac{1}{y-2}$.]