MAA 3200 - INTROD TO ADVANCED MATH FLORIDA INT'L UNIV.
TEST #2 - SPRING 2021 TIME: 75 min.

Answer all 6 questions. ***No calculators, notes, or on-line stuff are allowed.*** *An unjustified answer will receive little or no credit.* ***Draw a line to separate each of your 6 solutions to the 6 questions.***
(Double check the solutions that you send as a pdf file – to ensure it contains everything.)

(15) 1. (a) Define what is a *finite sequence* ***s*** and define what is a *subsequence* of ***s***.
(b) Define what it means for $f : A \to B$ to be *injective*. Prove that if $f : A \to B$ and $g : B \to C$ are both injective, then $g \circ f : A \to C$ is also injective.

(20) 2. (a) Let $g : X \to Y$ be a function and suppose that $A, B \subseteq X$ and $C, D \subseteq Y$. *Define* what is $g[A]$ and define what is $g^{-1}[C]$.
(b) Is it always true that $g[A] - g[B] \subseteq g[A-B]$?
(c) Is it always true that $g^{-1}[C] \cap g^{-1}[D] \subseteq g^{-1}[C \cap D]$?

(15) 3. (a) Write down the *Second (Strong) Principle of Mathematical Induction* for $\mathbb{N}$.
(b) Prove that $(\forall n \in \mathbb{N})$ [$5^{n+2} + 6^{2n+1}$ is an integer-multiple of 31].

(15) 4. (a) Define what it means for A to be *denumerable* and for B to be *countable*.
(b) Prove that $\mathbb{N} \times \mathbb{Z}^+$ is a denumerable set. [*If you claim that a function is a bijection, then you must prove that the function is indeed a bijection.*]

(20) 5. (a) Define what is a *convergent* sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ of real numbers.
Suppose $\langle a_n \rangle_{n \in \mathbb{N}}$ converges to A and $\langle b_n \rangle_{n \in \mathbb{N}}$ converges to B. Prove that
(b) $\langle 3a_n \rangle_{n \in \mathbb{N}}$ converges to 3A, and (c) $\langle b_n - a_n \rangle_{n \in \mathbb{N}}$ converges to B–A.

(15) 6. (a) Define what is a *bounded sequence* $\langle b_n \rangle_{n \in \mathbb{N}}$ sequence of real numbers. and what is a *Cauchy sequence* $\langle c_n \rangle_{n \in \mathbb{N}}$ of real numbers.
(b) Prove that if $\langle c_n \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence, then it is a *bounded sequence*.

$\in \ \forall \ \exists \ \Delta \ \oplus \subseteq \not\in \subset \rightarrow \neg \neq \infty \ \varnothing \equiv \approx \leftrightarrow \ \times \ \aleph \ \surd \ \nabla$ □ $\cong \perp \ \pm \geq \leq {}^\circ \uparrow \downarrow \perp - \cup \cap \mathbb{R} \ \mathbb{Z} \ \langle \ \rangle \ \mathbb{N}$

#1(a) A finite sequence $\underset{\sim}{s}$ is any function $\underset{\sim}{s}$ with $dom(\underset{\sim}{s}) = \mathbb{N}_k$ for some $k \in \mathbb{N}$. A subsequence of $\underset{\sim}{s}$ is any ordered pair $\langle \underset{\sim}{s}, \underset{\sim}{t} \rangle$ where $t: \mathbb{N}_\ell \to \mathbb{N}_k$ is an strictly increasing function and $dom(\underset{\sim}{s}) = \mathbb{N}_k$.

(b) $f: A \to B$ is injective if $(\forall a_1, a_2 \in A)\,[f(a_1) = f(a_2) \to (a_1 = a_2)]$
Suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. Since g is injective, we must have $f(a_1) = f(a_2)$. And since f is injective, we then get $a_1 = a_2$. So $(g \circ f)(a_1) = (g \circ f)(a_2) \to a_1 = a_2$. Hence $g \circ f : A \to C$ is injective.

#2 (a) $g[A] = \{g(a) : a \in A\}$ and $g^{-1}[C] = \{x \in X : g(x) \in C\}$.

2 (b) Let $y \in g[A] - g[B]$. Then $y \in g[A]$ and $y \notin g[B]$. So we can find an $a \in A$ such that $g(a) = y$. Now if $a \in B$, then $g(a) = y \in B$ which contradicts $y \notin B$. Hence $a \notin B$. $\therefore$ $a \in A - B$. Thus $y \in g[A-B]$ because $a \in A - B$. Hence $g[A] - g[B] \subseteq g[A-B]$.

2 (c) Let $x \in g^{-1}[C] \cap g^{-1}[D]$. Then $x \in g^{-1}[C]$ and $x \in g^{-1}[D]$. So $g(x) \in C$ and $g(x) \in D$. $\therefore$ $g(x) \in C \cap D$. $\therefore$ $x \in g^{-1}[C \cap D]$. Hence $g^{-1}[C] \cap g^{-1}[D] \subseteq g^{-1}[C \cap D]$.

#3 (a) Let $P(n)$ be a first order formula with free variable n. Then $\{P(0) \wedge (\forall n \in \mathbb{N})[P(0) \wedge P(1) \wedge \cdots \wedge P(n) \to P(n+1)]\} \Rightarrow (\forall n \in \mathbb{N})\, P(n)$.

3(b) Let $P(n)$ be the formula $(\exists k \in \mathbb{N})[5^{n+2} + 6^{2n+1} = 31k]$.
Since $5^{0+2} + 6^{2(0)+1} = 25 + 6 = 31(1)$, $P(0)$ is true. Now suppose $P(n)$ is true. Then $5^{n+2} + 6^{2n+1} = 31k$ for some $k \in \mathbb{N}$.
So $5^{(n+1)+2} + 6^{2(n+1)+1} = 5 \cdot 5^{n+2} + 5 \cdot 6^{2n+1} + 31 \cdot 6^{2n+1}$
$= 5(5^{n+2} + 6^{2n+1}) + 31 \cdot 6^{2n+1} = (5k + 6^{2n+1})(31)$
So $(\forall n \in \mathbb{N})[P(n) \to P(n+1)]$. Hence $P(n)$ is true for all $n \in \mathbb{N}$.

#5 (a) $\langle a_n \rangle_{n \in \mathbb{N}}$ is convergent if $(\exists L \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)[|a_n - L| < \varepsilon]$.

5(b) Suppose $\langle a_n \rangle$ converges to A. Let $\varepsilon > 0$ be given. Then $\varepsilon/3 > 0$. So we can find an $N \in \mathbb{N}$ such that $(\forall n \geq N)$ $|a_n - A| < \varepsilon/3$. So $(\forall n \geq N)$, $|3a_n - 3A| = 3|a_n - A| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$.

Hence $(\forall \varepsilon>0)(\exists N\in\mathbb{N})(\forall n\geq N)(|3a_n-3A|<\varepsilon)$. Thus $\langle 3a_n\rangle$ converges to $3A$.

5(c) Suppose $\langle a_n\rangle$ conv. to A & $\langle b_n\rangle$ conv. to B. Let $\varepsilon>0$ be given. Then $\varepsilon/2>0$. So we can find $N_1, N_2\in\mathbb{N}$ such that $(\forall n\geq N_1)\ (|a_n-A|<\varepsilon/2)$ & $(\forall n\geq N_1)(|b_n-B|<\varepsilon/2)$.
Let $N=\max\{N_1,N_2\}$. Then for all $n\geq N$,

$$|(b_n-a_n)-(B-A)| = |(b_n-B)+(A-a_n)| \leq |b_n-B|+|A-a_n|$$
$$= |b_n-B|+|a_n-A| < \varepsilon/2+\varepsilon/2=\varepsilon$$

$\therefore (\forall\varepsilon>0)(\exists N)(\forall n\geq N)[|(b_n-a_n)-(B-A)|<\varepsilon]$.
Hence $\langle b_n-a_n\rangle$ converges to $B-A$.

#4(a) The set A is denumerable if there exists a bijection $f: A\to\mathbb{N}$.
The set B is countable if there exists an injection $g: A\to\mathbb{N}$.
(We can also say B is countable if B is denumerable or B is finite)

4(b) Let $f:\mathbb{N}\times\mathbb{Z}^+\to\mathbb{N}$ be defined by $f(k,\ell)=2^k(2\ell-1)-1$.
Now suppose $f(k_1,\ell_1)=f(k_2,\ell_2)$. Then $2^{k_1}(2\ell_1-1)-1=2^{k_2}(2\ell_2-1)-1$
So $2^{k_1}(2\ell_1-1)=2^{k_2}(2\ell_2-1) \Rightarrow 2^{k_1}=2^{k_2}$ because $2\ell_1-1$ & $2\ell_2-1$ are both odd.
So $k_1=k_2$. And thus $2\ell_1-1=2\ell_2-1\Rightarrow \ell_1=\ell_2$. $\therefore f$ is injective.
Now take any $n\in\mathbb{N}$. Then $n+1$ can be expressed in the form $n+1=2^k(2\ell-1)$. For example $71+1=72=2^3\cdot[2(5)-1]$
So $f(k,\ell)=2^k\cdot(2\ell-1)-1=(n+1)-1=n$. Hence f is surjective.
Thus f is a bijection and so $\mathbb{N}\times\mathbb{Z}^+$ is a denumerable set.

#6(a) An infinite sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ is <u>bounded</u> if $(\exists U,L\in\mathbb{R})(\forall n\in\mathbb{N})$ $L\leq a_n\leq U$. $\langle c_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence if $(\forall\varepsilon>0)(\exists N\in\mathbb{N})$ $(\forall m,n\geq N)(|c_m-c_n|<\varepsilon)$.

6(b) Suppose $\langle c_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers.
Let $\varepsilon=1$. Then we can find an $N\in\mathbb{N}$ such that for all $m,n\geq N$ $|c_m-c_n|<1$. So $(\forall n\geq N)\ |c_N-c_n|<1$ in particular.
$\therefore (\forall n\geq N)\ [-1<c_n-c_N<1]$, i.e., $(c_N)-1<c_n<(c_N)+1$
Let $L=\min\{c_1,\ldots,c_{N-1},(c_N)-1\}$ & $U=\max\{c_1,c_2,\ldots,c_{N-1},(c_N)+1\}$
Then $(\forall n\in\mathbb{N})\ [L\leq c_n\leq U]$. So $\langle c_n\rangle$ is bounded.