

Ch. 1 - The Logic used in Mathematics

(1)

§0. Introduction:

Mathematics is, in a certain sense, the study of Formal Deductive Systems (FDS). An. FDS, S consists of an alphabet, $L(S)$, from which certain kinds of strings (called formulas) are made. Some of these formulas (which appear to be self-evident) are selected and called Axioms, $A(S)$. There are two parts of the alphabet $L(S)$ - the logical part and the mathematical (or proper) part. The Axioms $A(S)$ are also divided into two parts - the logical axioms & the mathematical (or proper) axioms. Finally there are rules of inference: $R(S)$ (which appear to be self-evident) which tells us how to derive one formula from one or more formulas. Thus an FDS $S = \langle L(S), A(S), R(S) \rangle$ - the alphabet $L(S)$ & the Axioms $A(S)$ both have a logical part and a mathematical part, but the rules of inference only has a logical part.

The logic used in the FDS's (for any part of mathematics) is called First Order Logic with Equality (FOLWE). So if we are studying Number Theory, then we will have as the alphabet the following:

logical part of $L(S)$ = {variables, equality sign, connectives, quantifiers, parentheses, comma}

The number theory part of L(S) consists of (2)
the constant symbols 0 & 1, the relation symbol
< and the function symbols S, +, & ". From
this alphabet we make up formulas and some
of these formulas (which deals only with FOLWE)
are designated as logical axioms. Some
other formulas which deal specifically with
properties of natural numbers are selected
and designated as number-theoretic axioms.
Finally the rules of inference will just be
Modus Ponens (MP) & Generalization (GEN) (but
we can include more rule of inf. if we wish to).

If we want to study Set Theory everything
will be pretty much the same as in number
theory, except that the Set Theory part of the alphabet
will consist of only the constant symbol \emptyset & the
relation symbol \in - and the Set-Theoretic
axioms will replace the number-theoretic axioms.
The rules of inference will again be MP & GEN
(whatever these might mean).

theoretically

Mathematics, consists of deriving (proving)
interesting formulas in various interesting
F.D.S's. Now what I have said here is a
brief summary of the Axiomatic Method used
in Mathematics - but here and there I had to
lie a little, because the subject is intrinsically
difficult. However, this is not how we actually
do mathematics - for if we did, we would,

(3)

very quickly, become "completely mad." Mathematics is a human activity and is usually conducted in a natural human language (such as English) — but we supplement the nat. language with mathematical symbols and mathematical terminology (abbreviations) to make things more precise, more manageable, and more comprehensible. (Yes, you read that correctly — more understandable!)

There is a course called MHF 4302 (Mathematical Logic) in which everything is done as we outlined here, but we have only a small Chapter 1 to do all this in order to get to the normal way mathematics is practised. So we will have to cheat a little. We will introduce a simple part of FOLWE called Propositional Logic because we can easily determine anything in Propositional Logic by using truth-tables. Then we will use this Propositional logic to help us understand (to a certain extent) First Order Logic with Equality (or Predicate Logic). We will not do things by using Formal Deductive Systems — but we will appeal to common sense and mathematical experience to guide us. Only when we are unsure will we revert to truth-tables & in MHF 4302 we, of course, have to revert to the Axioms.

(4)

§1. Propositional logic.

The alphabet of Propositional Logic consists of the connectives: $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$; parentheses & an infinite set of statement symbols (or basic propositions): $P, Q, R, P_1, P_2, P_3 \dots$.

Def. The formulas (or general propositions) of Propositional logic are defined recursively as follows:

- all statement symbols are formulas of Prop. logic
- if A and B are formulas of Prop. logic then so are $(\neg A)$, $(A \vee B)$, $(A \wedge B)$, $(A \rightarrow B)$ and $A \leftrightarrow B$.

Ex. 1 (a) P, Q, R, P_5 , and P_9 are all formulas.

(b) $(\neg P)$, $(\neg(\neg Q))$, $(P \wedge Q)$, $(Q \vee P)$, $(P \rightarrow Q)$, $(P \leftrightarrow R)$ and $(\neg(P \wedge Q))$

are all formulas.

(c) $(\neg P Q)$, $(\neg P \wedge)$, and $(P \rightarrow Q R)$ are not formulas.

We usually leave out the outer-most pair of parentheses because it saves us time and help us to avoid clutters.

Now in order to understand what a formula says we must interpret each of the symbols of a formula. The statement symbols stands for definitive declarative sentences (which can only be true or false). The connectives are interpreted by using truth tables and the parentheses are used to tell us the order in which the connectives were used. So let us look in more detail at the connectives.

(5)

Suppose P is a statement symbol. Then P could be true or P could be false. Now if P is true we insist that $\neg P$ must be false & if P is false we insist that $\neg P$ must be true. This gives us the interpretation of " \neg " even though we don't know (unless we are told which sentence to take as P) whether P is true or false. " \neg " is thus interpreted as the negation (or opposite) of P . We can summarise this information in a table (called a truth-table) as follows.

Def of \neg : $P \quad (\neg P)$

T	F
F	T

Now if we say what we intend as the interpretation for \vee , \wedge , \rightarrow and \leftrightarrow most logically sentient beings will arrive at the following truth tables.

" \vee " stands for or (disjunction)

" \wedge " stands for and (conjunction)

" \rightarrow " stands for conditional (implies or sufficient for)

" \leftrightarrow " stands for bi-conditional (if and only if),

Def of the other four connectives:

P	Q	$P \vee Q$	$P \wedge Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	T	F
F	F	F	F	T	T

(6)

If we look carefully at the truth table for $(P \rightarrow Q)$, we will find that when P is false & Q is true, we say that $P \rightarrow Q$ is true. Not every layperson might agree with this (but all mathematicians agree) — so to avoid any controversy, we usually say that this is our intended way of interpreting " \rightarrow " and stop any further discourse. Why? Because if we interpret " \rightarrow " this way, then everything makes sense (for us) and we get beautiful results in logic. We should also mention that "or" is sometimes interpreted differently (as exclusive or) in English — so we again insist that " \vee " is to be interpreted as inclusive or ($A \vee B$ means A or B or both.)

: There are other connectives such as:

\oplus (xor)	<u>exclusive or</u> (exactly one),
\leftarrow (isimp)	<u>reverse conditional</u> (is necessary for)
\downarrow (nor)	not $(P \vee Q)$ (Peirce's arrow)
\uparrow (nand)	not $(P \wedge Q)$ (Sheffer's arrow)
$\not\rightarrow$ (nimp)	not $(A \rightarrow B)$, & $\not\leftarrow$ (nisimp) not $(B \leftarrow A)$

but we will hardly use any of them in this course.

P	Q	$P \oplus Q$	$P \leftarrow Q$	$P \downarrow Q$	$P \uparrow Q$	$P \# Q$	$P \not\rightarrow Q$
T	T	F	T	F	F	F	F
T	F	T	T	F	T	T	F
F	T	T	F	F	T	F	T
F	F	F	T	T	T	F	F

(7)

Def. A formula A of Propositional Logic (or a general proposition) is a tautology if the truth-value of A is always T (no matter what may be the truth values of the constituent statement symbols of A).

Def. A formula B of Propositional Logic is a contradiction if the truth-value of B is always F (no matter what the truth values of the constituent statement letters of B may be.)

Ex. 2

(a) $P \vee (\neg P)$, $(P \rightarrow P)$, and $P \rightarrow (P \vee Q)$ are all tautologies

(b) $P \wedge (\neg P)$, $\neg(P \rightarrow P)$, and $(P \rightarrow Q) \wedge (P \wedge \neg Q)$ are all contradictions

P	$P \vee (\neg P)$
T	(T) F
F	(T) T

tautology

P	$P \wedge (\neg P)$
T	(F) F
F	(F) T

contradiction

(c) $P \wedge (\neg Q)$, $P \rightarrow (\neg Q)$, and $(\neg P) \leftrightarrow (\neg Q)$ are neither tautologies nor contradictions.

So we now have a pretty good idea of what are the formulas of Propositional Logic — and we also know that some of these formulas are very special and are called tautologies. But what is the point of all these formulas? What do we do with them? What are the main questions in Prop. Logic?

§.2 Logically valid reasoning in Propositional Logic

Well, there are two basic questions that we can ask in Propositional Logic.

- Qn (a) Does the formula A logically imply the formula B?
 Qn (b) Is the formula A logically equivalent to B?

What do "logically imply" & "logically equivalent" mean?

- Def.(a) A logically implies B if whenever A has truth-value T, then B has truth-value T.
(b) A is logically equivalent to B if A has truth-value T if and only if B has truth value T.

Notation We will use the symbols " \Rightarrow " & " \Leftrightarrow " to indicate logical implication & logical equivalence. So

$A \Rightarrow B$ abbreviates "A logically implies B"
 $A \Leftrightarrow B$ abbreviates "A is logically equivalent to B".

Ex.3

- (a) $P \Rightarrow P$, $(P \wedge Q) \Rightarrow P$, $(\neg P) \Rightarrow (P \Rightarrow Q)$.
 (b) $P \Leftrightarrow P$, $(P \wedge Q) \Leftrightarrow (Q \wedge P)$, $\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$.
 (c) $(\neg P) \Leftrightarrow (P \Rightarrow Q)$, $P \not\Rightarrow (\neg P)$, $(Q \vee P) \not\Rightarrow (P \wedge Q)$.

a(iii)

		$(\neg P) \Rightarrow P \Rightarrow Q$	
P	Q	$(\neg P) \Rightarrow P \Rightarrow Q$	
T	T	F	T
T	F	F	F
* F	T	T	T
* F	F	T	T

: only the 3rd & 4th rows
are relevant here

		$\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$	
P	Q	$\neg(P \vee Q) \Leftrightarrow (\neg P) \wedge (\neg Q)$	
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	F

All 4 rows are relevant here.

(9)

Theorem 1

- (a) $(A \Rightarrow B)$ if and only if $(A \rightarrow B)$ is a tautology
 (b) $(A \Leftrightarrow B)$ if and only if $(A \leftrightarrow B)$ is a tautology.

Note. Even though " \Rightarrow " (logically implies) and \rightarrow (implies) seem to be basically the same thing, they are not. $(A \rightarrow B)$ is a formula of Propositional Logic and one should think of it as an expression in Algebra, such as $(x+y)$. So $A \rightarrow B$ is not stating anything, it is just a "silent" expression. $(A \Rightarrow B)$, on the other hand is not a formula of Propositional Logic because " \Rightarrow " is not a part of the alphabet of Prop. Logic. " \Rightarrow " is a symbol from the meta-language, which in this case is English language plus mathematical symbols, that is being used to understand Propositional logic.) Also $(A \Rightarrow B)$ is actually saying something, so it is more like the algebraic statement $x+y \geq x-y$. $(A \Rightarrow B)$ is either true or it is false. On the other hand, $(A \rightarrow B)$ has an interpretation which depends on the truth values of its constituent statement symbols. $x+y$ or x^2-y^2 are algebraic expressions which do not say anything.

Similarly $A \Leftrightarrow B$ is a formula of Prop. logic and it is also like an algebraic expression; while $A \Leftrightarrow B$ is not a formula of Prop. logic and it is like the algebraic expression $x^2-y^2 \equiv (x-y)(x+y)$.

$(A \Leftrightarrow B)$ must be true or false, while $(A \leftrightarrow B)$ is (10)
 an expression that has an interpretation but
 is not saying anything. $x^2 - y^2$ is an expression
 that you can interpret once you think what x
 and y are. In the same way, $A \leftrightarrow B$ can
 be interpreted once we think of its constituent
 statement symbols. So we can think of $A \leftrightarrow B$
 as an expression involving its statement letters
 and the connectives.

We can generalise Qu(a). as follows. Let $(A_i : i \in I)$
 be a set of formulas of Propositional Logic.
 $(A_i : i \in I)$ may be finite or infinite.

Qu.(c) Does $(A_i : i \in I)$ logically imply B , i.e., whenever
 all the A_i 's have truth value T, does B have truth value T?

Notation

We will use the symbol \models as follows

$(A_i : i \in I) \models B$ abbrev. $(A_i : i \in I)$ logically implies B .
 We also sometimes say B is a logical consequence
 of $(A_i : i \in I)$

Note: If $(A_i : i \in I)$ is finite, say $I = \{1, 2, \dots, n\}$, then
 $(A_i : i \in \{1, 2, \dots, n\}) \models B$ just says that
 $\{A_1, \dots, A_n\} \models B$, i.e., $(A_1 \wedge A_2 \wedge \dots \wedge A_n) \Rightarrow B$.

For a long time, logic was concerned with
 questions of the form:

Major Premise	A_1
Minor Premise	A_2
\therefore Conclusion	$\therefore B$

(11)

Ex.4 (a) If I am rich, I am happy. (b) $R \rightarrow H$

I am not happy. $\neg H$

\therefore I am not rich. $\therefore \neg R$

Let $R = \text{I am rich}$ & $H = \text{I am happy}$. Then we want to know if the symbolic argument in (b) is valid. To determine if this is so, we need to check if $[(R \rightarrow H) \wedge (\neg H)] \Rightarrow (\neg R)$. To check this, we need to check if $[(R \rightarrow H) \wedge (\neg H)] \Rightarrow (\neg R)$ is a tautology.

		$[(R \rightarrow H) \wedge (\neg H)] \Rightarrow (\neg R)$				
R	H	T	F	F	T	F
T	T				T	
T	F	F	F	T	T	F
F	T	T	F	F	T	T
F	F	T	T	T	T	T

tautology

So the argument is logically valid.

Ex.5 (a) If you have a good spouse, you will be happy
 If you don't have a good spouse, you'll become a philosopher
You are not a philosopher.
 \therefore you are happy. Is this argument valid?

Let $G = \text{you have a good spouse}$, $H = \text{you are happy}$
 and $P = \text{you are a philosopher}$. (b) $G \rightarrow H$
 Then the argument becomes (b). $\frac{\neg G \rightarrow P}{\therefore H}$

To know if the argument is valid, we need to check if $[(G \rightarrow H) \wedge (\neg G \rightarrow P)] \Rightarrow H$ is a tautology.
 This looks like hard work! And it is!

G	H	P	$\{\{G \rightarrow H\} \wedge (\neg G \rightarrow P)\} \wedge (\neg P)$	$\rightarrow H$	(12)
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	F	P
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	F
F	F	F	T	F	E

So the argument is indeed valid. Is there an easier way? We will soon see that there is.

§3. Laws & Rules of inference of Propositional Logic

By using truth tables, we can justify the following laws of propositional logic.

- not so
obvious*

1.	Double negation law: $\neg(\neg A) \Leftrightarrow A$
2.	Idempotency laws: (a) $(A \wedge A) \Leftrightarrow A$ (b) $(A \vee A) \Leftrightarrow A$
3.	Commutative laws: (a) $(A \wedge B) \Leftrightarrow (B \wedge A)$ (b) $(A \vee B) \Leftrightarrow (B \vee A)$
4.	Associative laws: (a) $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$ (b) $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$
5.	Distributive laws: (a) $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$ (b) $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$
6.	De Morgan's negation laws: (a) $\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$ (b) $\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$.

- not at all obvious
- one-way laws
- | | |
|--|---|
| 7. Conditional Law: $(A \rightarrow B) \Leftrightarrow (\neg A \vee B)$ | 8. Contrapositive law: $(A \rightarrow B) \Leftrightarrow (\neg B \rightarrow \neg A)$ |
| 9. Converse Conditional law: $(B \rightarrow A) \Leftrightarrow ((\neg A) \rightarrow (\neg B))$ | 10. Exportation law: $((A \wedge B) \rightarrow C) \Leftrightarrow (A \rightarrow (B \rightarrow C))$ |
| 11. Hypothetical syllogism law: $((A \rightarrow B) \wedge (B \rightarrow C)) \Rightarrow (A \rightarrow C)$ | 12. Disjunctive syllogism law: $((A \vee B) \wedge \neg A) \Rightarrow B$ |
| 13. Modus Ponens law: $((A \rightarrow B) \wedge A) \Rightarrow B$ | 14. Modus Tollens law: $((A \rightarrow B) \wedge \neg B) \Rightarrow \neg A$. |

Def. A rule of inference in Propositional logic is any function that takes a finite number of formulas of Prop. Logic and produces another formula of Prop. Logic. (We usually require that the formula produced have truth value T whenever the input formulas have truth value T.)

Consider the Modus Ponens Law, $(A \rightarrow B) \wedge A \Rightarrow B$.

We say that from $(A \rightarrow B)$ and A , we can infer B and we write this as $(A \rightarrow B), A \vdash B$. This is the rule of inference known as Modus Ponens.

From each of the 14 laws of Prop. Logic given above we can get rules of inference - two each in the case of the first 10. So here are a few rules of inference.

- R 1. $(A \rightarrow B) \vdash (\neg A \vee B)$, $(\neg A \vee B) \vdash (A \rightarrow B)$ from law 7.
- R 2. MODUS PONENS rule: $(A \rightarrow B), A \vdash B$
- R 3. MODUS TOLLENS rule: $(A \rightarrow B), \neg B \vdash \neg A$
- R 4. DISJUNCTIVE SYLLOGISM rule: $(A \vee B), \neg A \vdash B$
- R 5. HYPOTHETICAL SYLLOGISM rule: $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$.
- R 6. PROOF BY CONTRADICTION: $\neg A \rightarrow B, \neg A \rightarrow \neg B \vdash A$.

Ex.6 Now let us look back at the symbolic argument in

$$\begin{array}{lll}
 5(b) \quad G \rightarrow H & \text{We can replace } \neg G \rightarrow P & G \rightarrow H \\
 & \neg G \rightarrow P & \neg P \rightarrow G \\
 & \neg P & \neg P \\
 \hline
 & \therefore H & \neg P
 \end{array}$$

Now from $\neg P \rightarrow G$ and $\neg P$ we can get G by Modus Ponens; and from $G \rightarrow H$ and G we can get H by Modus Ponens again. So the argument is indeed valid - and this is a lot quicker than any truth-table checking.

Ex.7 What is the negation (opposite) of the statement: If I am rich, then I am happy.

Let $R = \text{I am rich}$ and $H = \text{I am happy}$. Then the statement can be abbreviated as $R \rightarrow H$. So we need $\neg(R \rightarrow H)$ translated into understandable English. Now, "It is not the case that if I am rich, then I am happy," is not an adequate answer because this sheds no light and is not a very understandable sentence. So we will find an equivalent statement to $\neg(R \rightarrow H)$ which has an understandable English sentence.

$$\begin{aligned}
 \neg(R \rightarrow H) &\Leftrightarrow \neg[(\neg R) \vee H] \\
 &\Leftrightarrow \neg(\neg R) \wedge \neg(H) \Leftrightarrow R \wedge \neg(H).
 \end{aligned}$$

So the negation is, "I am rich and I am not happy."

Note: $\neg(R \rightarrow H)$ is not equivalent to any of the following $(\neg R) \rightarrow H$, $(\neg R) \rightarrow \neg H$, $R \rightarrow (\neg H)$, $H \rightarrow (\neg R)$, $(\neg H) \rightarrow \neg R$, $(\neg H) \rightarrow R$.

- Def A conditional formula is one of the form $(A \rightarrow B)$
- The converse of $(A \rightarrow B)$ is $(B \rightarrow A)$.
- The inverse of $(A \rightarrow B)$ is $(\neg A) \rightarrow (\neg B)$
- The contrapositive of $(A \rightarrow B)$ is $(\neg B) \rightarrow (\neg A)$,

Note: $(A \rightarrow B) \Leftrightarrow [(\neg B) \rightarrow (\neg A)]$ and $(B \rightarrow A) \Leftrightarrow [(\neg A) \rightarrow (\neg B)]$

We can define generalised conjunctions and generalised disjunctions because of the associative laws for Prop. Logic. Let A_1, \dots, A_n be formulas

- Def. We define the conjunction $\bigwedge_{i=1}^n A_i$ recursively as follows.
- $\bigwedge_{i=1}^1 A_i$ to be A_1 ,
 - $\bigwedge_{i=1}^{n+1} A_i$ to be $(\bigwedge_{i=1}^n A_i) \wedge A_{n+1}$.
- We usually write $\bigwedge_{i=1}^n A_i$ as $(A_1 \wedge A_2 \wedge \dots \wedge A_n)$

- Def. We also define the disjunction $\bigvee_{i=1}^n A_i$ recursively as follows.
- $\bigvee_{i=1}^1 A_i$ to be A_1 ,
 - $\bigvee_{i=1}^{n+1} A_i$ to be $(\bigvee_{i=1}^n A_i) \vee A_{n+1}$.
- We usually write $\bigvee_{i=1}^n A_i$ as $(A_1 \vee A_2 \vee \dots \vee A_n)$

The following equivalence laws can be easily verified by using Mathematical Induction.

1. Generalised De Morgan's laws:

$$(a) \neg \left(\bigwedge_{i=1}^n A_i \right) \Leftrightarrow \bigvee_{i=1}^n (\neg A_i) \quad (b) \neg \left(\bigvee_{i=1}^n A_i \right) \Leftrightarrow \bigwedge_{i=1}^n (\neg A_i)$$

2. Generalised Distributive laws:

$$(a) A \wedge \left(\bigvee_{i=1}^n B_i \right) \Leftrightarrow \bigvee_{i=1}^n (A \wedge B_i) \quad (b) A \vee \left(\bigwedge_{i=1}^n B_i \right) \Leftrightarrow \bigwedge_{i=1}^n (A \vee B_i)$$

§4. Predicate (or First Order) logic with Equality

16

First Order Logic with Equality is a generalisation of Propositional Logic and we use the term "with equality" to emphasize that equality is included. It is universally accepted as the appropriate logic to use in all of math.

The alphabet of a First Order Logic consists of

(f) In addition there may be zero or more

- (i) constant symbols: $a, b, c, c_1, c_2, c_3 \dots$
 - (ii) function symbols: $f, g, h, f_1, f_2, f_3 \dots$ of various arities, and
 - (iii) relation symbols: $P, Q, R, R_1, R_2, R_3, \dots$ of various arities

These are called the paper symbols of the alphabet.

The arity of a function is the number of inputs it requires (binary = arity 2) and the arity of a relation is the number of variables it involves (ternary = arity 3).

In order to define the formulas of First Order Logic we need to first define what are terms of First order Logic. Then we will define what are atomic formulas. And finally we will define what are (general) formulas.

Def. The terms of first order logic are defined recursively as follows:

- (a) Any variable or constant symbol is a term.
- (b) If f is a function symbol of arity n , and t_1, \dots, t_n are terms then $f(t_1, \dots, t_n)$ is also a term.

Ex.1 Let us take a first order logic with the proper part of the alphabet being $\{0, 1, s, a, m, <\}$. Then

- (a) $0, 1, s(1), \underbrace{m(s(0), 1)}_{=s(0) \cdot 1}, \underbrace{a(s(1), 1)}_{=s(0)+1}$ are all terms
- (b) $x, y, s(x), \underbrace{m(x, y)}_{=x \cdot y} \text{ and } \underbrace{a(s(x), 1)}_{=s(x)+1}$ are all terms.

Def. The atomic formulas of first order logic are defined as follows:

- (a) If t_1 and t_2 are terms then $(t_1 = t_2)$ is an atomic formula.
- (b) If t_1, \dots, t_n are terms and R is a relation symbol of arity n , then $R(t_1, \dots, t_n)$ is also an atomic formula.

Ex.2 Let us take the same alphabet (of Arithmetic) that was used in Ex.1. Then

- (a) $s(x) = a(x, 1)$ is an atomic formula.
- (b) $m(x, 1) = a(x, 0)$ is an " "
- (c) $(0 < 1), (x < 0)$ are both " "
- (d) $s(x) < s[a(x, 1)]$ is an atomic formula.

Def. Finally the (general) formulas of first order logic are defined recursively as follows:

- (a) all atomic formulas are formulas of first order logic.
- (b) If A and B are formulas of first order Logic and x is a variable, then
 $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$,
 $(\forall x A)$, and $(\exists x A)$ are also formulas.

Ex.3 Again, let us use the same alphabet (of Arithmetic) from Ex. 1. Then

$\neg(0=1)$, $\neg(1 < 0)$, $(x < 0) \vee (0 < x)$,
 $(0 < x) \rightarrow (y < a(x, y))$, $(\exists x (x < 0))$, and
 $\neg(\exists x)(\forall y (y < x))$ are all formulas.

Now, just like in Propositional Logic, we must interpret each of the symbols in a formula to understand what it means. For this we need the concept of a structure

Def. A structure that is compatible with the alphabet of a First Order Logic consists of a non-empty domain D

- (a) particular elements a_D for each of the constant symbols
- (b) particular functions for each of the function symbols
- & (c) particular relations for each of the relation symbols.

Ex.4 So a compatible structure for the alphabet of Arithmetic is $\langle \text{IN}; 0, 1, s, +, \cdot, <, \rangle$ where

(19)

Ex. 4 Here $N = \{0, 1, 2, 3, \dots\}$ is the domain and we interpret 0 & 1 as the numbers 0 & 1 in N , "s" by the function $s(x) = x+1$, "a & m" by the functions $a(x, y) = x+y$ & $m(x, y) = x \cdot y$ and " $<$ " by the "less than" relation in N .

The connectives $\neg, \wedge, \vee, \rightarrow$, and \leftrightarrow are interpreted the same way as in Prop. Logic, " $=$ " is the usual notion of equality and " $\forall x$ " means "for all ..." while " $\exists x$ " means "there is an x such that ...". The variables x, y, z , etc. stand for unknown quantities which range over the whole domain.

Now if we take another compatible structure such as $\langle \mathbb{Z}; 0, 1, s, +, \cdot, < \rangle$ then the meaning (and truth-hood) of a formula of arithmetic will be different. We will say that a formula is true, if it is true in all possible interpretations, and, ^{then} call it a tautology (if this is so).

Def. Let A and B be formulas of First Order Logic.

- (a) We say that A logically implies B & write $A \Rightarrow B$ if the formula $(A \rightarrow B)$ is true in all possible interpretations, (i.e. if $A \rightarrow B$ is a tautology)
- (b) We say that A is logically equivalent to B & write $A \Leftrightarrow B$, if the formula $A \leftrightarrow B$ is true in all possible interpretations, (i.e. if $A \leftrightarrow B$ is a tautology)

Def (c) Let $\langle A_i : i \in I \rangle$ be a set of formulas. We say

$\langle A_i : i \in I \rangle$ logically implies B (and that B is a logical consequence of $\langle A_i : i \in I \rangle$) and write $\langle A_i : i \in I \rangle \models B$ if B is true in all interpretations in which all the A_i 's are true.

Ex. 5 Indicate which of the following formulas are true in (i) $\langle \mathbb{N}, < \rangle$ and (ii) $\langle \mathbb{Z}, < \rangle$.

- (a) $(\forall x)(\exists y)(x < y)$
- (b) $(\exists x)(\forall y)(x < y \vee x = y)$
- (c) $(\forall x)(\exists y)(y < x)$
- (d) $(\exists x)(\forall y)(y < x \vee x = y)$

(a) This formula says, "For all x there is a y with $x < y$ ". Now if we are given any x , we just have to take y to be $x+1$. Then we see that (a) is true in both $\langle \mathbb{N}, < \rangle$ & $\langle \mathbb{Z}, < \rangle$.

(b) This formula says "This says that there is an x such that for all y , $x \leq y$ ".

If we take x to be 0 in $\langle \mathbb{N}, < \rangle$, then we see that for all $0 \leq x$. So (b) is true in $\langle \mathbb{N}, < \rangle$. But in $\langle \mathbb{Z}, < \rangle$, no matter what x you take, we can always take y to be $x-1$ and then we won't have $x \leq y$.

So (b) is false in $\langle \mathbb{Z}, < \rangle$. Basically, (b) is saying that there is a smallest element - and this happens to be true in $\langle \mathbb{N}, < \rangle$ but false in $\langle \mathbb{Z}, < \rangle$.

(2i)

- (c) This formula says: "For all x , there is a y with $y < x$ ".
 In $\langle \mathbb{N}, < \rangle$ if we take x to be 0, then there is no y with $y < 0$. So (c) is false in $\langle \mathbb{N}, < \rangle$. However, in $\langle \mathbb{Z}, < \rangle$ — if we are given any x all we have to do is take y to be $x-1$ and then we will have $y < x-1$. So (c) is true in $\langle \mathbb{Z}, < \rangle$.

- (d) This formula says: "There is an x such that for all y , $y \leq x$ ". In otherwords, it says that there is a largest element. Since $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$ have no largest element, (d) is false in both $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$.

Note: Since $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$ were both infinite structures, we could not use truth-tables to help us determine whether or not a formula was true in $\langle \mathbb{N}, < \rangle$ or $\langle \mathbb{Z}, < \rangle$. So how in the world are we going to determine whether or not a formula is a tautology (i.e., whether or not it is true in all possible interpretations)?

Fortunately, we have the axiomatic method. We can find a list of formulas which are clearly true in all interpretations and call these axioms for First Order Logic — and then use the rules of inference MODUS PONENS & GENERALIZATION to arrive at all possible tautologies. But this is the subject matter of a Higher Course called Math. Logic.

85. Laws & Rules of inference of First Order Logic

We shall not use the Axiomatic Method to determine whether or not a formula of First Order Logic is a tautology. But if we knew how to do this, we could answer any other question such as "Does $A \Rightarrow B?$ " or "Is $A \Leftrightarrow B?$ " in First Order Logic. Instead, we will just give a few more laws & rules of inference to help us figure out these kinds of questions. Before we do this, however, let us look at a simpler problem.

Ex.1 What is the negation of the statement: Every cloud has a silver lining.

Ans: Let $C(x)$ be " x is a cloud" and $B(x)$ be " x has a silver lining". Then the statement says: $(\forall x)[C(x) \rightarrow B(x)]$. So our answer will be $\neg(\forall x)[C(x) \rightarrow B(x)]$ translated into ordinary English.

Now common sense can help us to see that

$$\begin{aligned} \neg(\forall x)[C(x) \rightarrow B(x)] &\Leftrightarrow (\exists x)\neg[C(x) \rightarrow B(x)] \\ &\Leftrightarrow (\exists x)\neg[\neg C(x) \vee B(x)] \\ &\text{from the Cond. law from Prop. Logic} \\ &\Leftrightarrow (\exists x)[\neg\neg C(x) \wedge \neg B(x)] \\ &\text{by De Morgan's law} \\ &\Leftrightarrow (\exists x)[C(x) \wedge \neg B(x)] \end{aligned}$$

which says, "There is a cloud which does not have a silver lining."

(23)

Ex.2 Consider the following argument:

We will all pass MAA 3200 if we do all the HW
 Not all of us passed MAA 3200.

∴ Some of us did not do all the H.W.
 Is this argument logically valid?

Ans: Let $H(x) = x \text{ did all the H.W. in MAA 3200}$ and
 $P(x) = x \text{ passed MAA 3200}$.

The argument says :
$$\frac{(\forall x)[H(x) \rightarrow P(x)]}{\neg(\forall x)[P(x)]}$$

 $\therefore (\exists x)[\neg H(x)]$

By Propositional Logic we can replace $H(x) \rightarrow P(x)$
 by $(\neg P(x)) \rightarrow (\neg H(x))$ and by our newly learnt
 common sense we can replace $\neg \forall x[P(x)]$
 by $(\exists x)[\neg P(x)]$. So the
 premises now become. $(\forall x)[\neg P(x) \rightarrow \neg H(x)]$
 $(\exists x)[\neg P(x)]$

Now since $(\exists x)[\neg P(x)]$, this means that we can
 find someone, call him c, such that $\neg P(c)$.
 But since $(\forall x)[\neg P(x) \rightarrow \neg H(x)]$, this means
 that $\neg P(c) \rightarrow \neg H(c)$. So we now have
 $\neg P(c) \rightarrow \neg H(c)$ and $\neg P(c)$. From Prop. Logic
 we can deduce that $\neg H(c)$ by Modus Ponens.
 Since we have $\neg H(c)$, we can now say
 $(\exists x)[\neg H(x)]$. So the argument is indeed
 logically valid.

And to do stuff like this again and again all we
 have to do is to add a few more laws & rules of inference.

(24)

To help us understand and think about " $\forall x$ " and " $\exists x$ " better - we can think of $(\forall x)R(x)$ as $\bigwedge_{x \in D} R(x)$ and $(\exists x)R(x)$ as $\bigvee_{x \in D} R(x)$. Now if D is infinite then these expressions are not formulas of Prop. Logic - but still it can help us to easily find the format of the appropriate laws.

Also
Valid
for
bounded
quantifiers

1. Quantifier negation laws:
 - (a) $\neg(\forall x)P(x) \Leftrightarrow (\exists x)[\neg P(x)]$ compare with the
 - (b) $\neg(\exists x)P(x) \Leftrightarrow (\forall x)[\neg P(x)]$. Generalized De Morgan's laws
2. Quantifier Commutative laws:
 - (a) $(\forall x)(\forall y)R(x,y) \Leftrightarrow (\forall y)(\forall x)R(x,y)$
 - (b) $(\exists x)(\exists y)R(x,y) \Leftrightarrow (\exists y)(\exists x)R(x,y)$
3. Quantifier Distributive laws:
 - a) $(\forall x)[P(x) \wedge Q(x)] \Leftrightarrow (\forall x)P(x) \wedge (\forall x)Q(x)$
 - b) $(\exists x)[P(x) \vee Q(x)] \Leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$

Remember " \forall " behaves like " \wedge " and " \exists " behaves like " \vee ".
4. One-way Quantifier laws:
 - a) $(\forall x)P(x) \vee (\forall x)Q(x) \Rightarrow (\forall x)[P(x) \vee Q(x)]$
 - b) $(\exists x)[P(x) \wedge Q(x)] \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$
 - c) $(\forall x)P(x) \Rightarrow (\exists x)P(x)$
 - d) $(\exists x)(\forall y)R(x,y) \Rightarrow (\forall y)(\exists x)R(x,y)$.

New rules of inference

- R1. Generalization Rule : If we have derived $P(x)$ for an arbitrary fixed x , deduce $(\forall x)P(x)$.
- R2. Particularization Rule : If we have derived $(\forall x)P(x)$, then deduce $P(c)$ for any fixed c .
- R3. Existence Rule : If we derive $\exists xP(x)$ then deduce $P(c)$ for some c

Other kinds of quantifiers.

Suppose we are thinking of a particular domain D and we have a special subset A of D and we want to say, "there is an $x \in A$ such that ..." or we want to say, "for all $x \in A$...". How can we do this. Well, we can introduce two new quantifiers called bounded (or restricted) quantifiers as follows:

- Def (a) $(\exists x \in A) P(x)$ abbreviates $(\exists x) [(x \in A) \wedge P(x)]$
 $(\forall x \in A) P(x)$ abbreviates $(\forall x) [x \in A \rightarrow P(x)]$

Why did we define $(\forall x \in A) P(x)$ differently from $(\exists x \in A) P(x)$? Well, we wanted to ensure that $\neg (\forall x \in A) P(x) \Leftrightarrow (\exists x \in A) [\neg P(x)]$ — so we were forced to define things as above. If we defined $(\forall x \in A) P(x)$ to be $(\forall x) [x \in A \wedge P(x)]$ then things would not have worked out.

Note: The first 3 quantifier laws hold for bounded (restricted) quantifiers — this was by design.

Finally we sometimes want to say, "there is exactly one x such that ..." For this we introduce the uniqueness quantifier as follows.

- Def $(\exists ! x) P(x)$ abbreviates $(\exists x) [P(x) \wedge (\forall y) \{ P(y) \rightarrow (y=x) \}]$

Ex. 3 (a) $(\exists ! x) (x \text{ is an even prime number})$

(b) $(\exists ! x) (\forall y) (x \text{ is an integer multiple of } y)$

(26)

Let $L(x, y)$ be " x loves y ". There are 10 possibilities

1. $(\forall x)(\forall y) L(x, y)$

Everyone loves everybody

$(\forall y)(\forall x) L(x, y)$

Everybody is loved by everyone

	a	b	c	d
a	o	o	o	o
b	o	o	o	o
c	o	o	o	o
d	o	o	o	o

2. $(\exists x)(\forall y) L(x, y)$

Someone loves everybody

	a	b	c	d
a				
b				
c	o	o	o	o
d				

3. $(\exists y)(\forall x) L(x, y)$

Somebody is loved by everyone

	a	b	c	d
a				
b				
c	o	o	o	o
d				

4. $(\forall x)(\exists y) L(x, y)$

Everyone loves somebody

	a	b	c	d
a				
b				
c				
d	o	o	o	o

5. $(\forall y)(\exists x) L(x, y)$

Everybody is loved by someone

	a	b	c	d
a				
b				
c	o	o	o	o
d				

6. $(\forall x) L(x, x)$

Everyone loves him or herself

	a	b	c	d
a	o	o	o	o
b	o			
c				
d				

7. $(\exists x) L(x, x)$

Someone loves him or herself

	a	b	c	d
a				
b				
c			o	
d				

8. $(\exists x)(\exists y) L(x, y)$

Someone loves somebody

$(\exists y)(\exists x) L(x, y)$

Somebody is loved by someone

	a	b	c	d
a				
b				
c			o	
d				

END

equiv.