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## Ch. 4 - Functions & their applications

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Introduction : The concept of a function is arguably one of the most important concept in mathematics and arose out after many years of efforts by several mathematicians. A function was thought to be a rule which produced a specific output from any given input in a specified set. Since the rule was usually expressed by a formula, a function became, inevitably, entwined with formulas. This is the way functions are still introduced in Elementary courses all the way up through Calculus and up to Differential Equations.

All this changed, however, with the work of Joseph Fourier in the early 1800's. Fourier handled functions by expanding them as infinite trigonometric series - and then later took any infinite trigonometric series to be a function. Naturally many mathematicians were skeptical and did not think that such things should be called functions. This was clarified a little later by Peter Dirichlet - and in the end it led to our modern concept of a function as a special kind of relation  $F$  in which any element of the domain of  $F$  can be related to only one element in the range of  $F$ .

Now there is more to a function than a view of it as a special relation - but this will be our foundation and starting point for understanding functions.

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### §1. Functions, injective functions, & Their inverses

Def: A function is a relation  $F$  such that

$$(\forall a \in \text{dom}(F)) [(a, b) \in F \wedge (a, c) \in F \rightarrow (b = c)].$$

In other words, a function is a special kind of relation in which an element in the domain is allowed to be related to only one element in the range.

Ex: Let  $F = \{(1, 5), (2, 6), (3, 7)\}$ ,  $G = \{(1, 4), (2, 4), (2, 5)\}$ ,  
 $H = \{(1, 2), (2, 2), (3, 8)\}$  &  $L = \{(1, 5), (5, 5), (5, 1)\}$ .

Then  $F$  &  $H$  are functions, ! .  $G$  is not a function because  $(2, 4) \in G \wedge (2, 5) \in G$  but  $2 \neq 5$ .  
Also  $L$  is not a function because  $(5, 5) \in L \wedge (5, 1) \in L$  but  $5 \neq 1$ . The empty set  $\emptyset$  is a function.

Note: Remember a function  $F$  is a relation - so  $F^{-1}$  is also a relation but it might not actually be a function. Also if  $F$  &  $G$  are functions then  $G \circ F$  is also already defined and  $G \circ F$  will be a relation - it might not turn out to be a function - but it actually is.

Now recall that if  $R$  was a relation, we usually write  $aRb$  to mean  $(a, b) \in R$ . Since a function  $F$  is also a relation we can write  $aFb$  to mean  $(a, b) \in F$  - but we don't usually do so unless we want to stress that  $F$  is a relation.

Notation: We will write  $F(a) = b$  to mean  $(a, b) \in F$ .

Note that since  $F$  is a function,  $F(a)$  can be only one value

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Prop.1: Suppose  $F$  &  $G$  are functions. Then  $G \circ F$  will also be a function and whenever  $a \in \text{dom}(G \circ F)$ ,  $(a, G(F(a))) \in G \circ F$ , i.e.,  $(G \circ F)(a) = G(F(a))$ .

Proof: Recall that  $(a, c) \in G \circ F$  if we can find  $b \in \text{dom}(G)$  such that  $(a, b) \in F$  &  $(b, c) \in G$ . Now in order to show that  $G \circ F$  is a function, we must show  
 $(\forall a \in \text{dom}(G \circ F)) [ (a, c_1) \in G \circ F \wedge (a, c_2) \in G \circ F \rightarrow (c_1 = c_2) ]$

So suppose  $(a, c_1) \in G \circ F$  &  $(a, c_2) \in G \circ F$ . Then we can find  $b_1, b_2 \in \text{dom}(G)$  such that  $(a, b_1) \in F \wedge (b_1, c_1) \in G$  and  $(a, b_2) \in F \wedge (b_2, c_2) \in G$ . So  $(a, b_1) \in F \wedge (a, b_2) \in F$ . Since  $F$  is a function, it follows that  $b_1 = b_2$ . So  $(b_1, c_1) \in G \wedge (b_1, c_2) \in G$  because  $(b_2, c_2) = (b_1, c_2)$ . Since  $G$  is also a function, it follows that  $c_1 = c_2$ . Hence

$$[(a, c_1) \in G \circ F \wedge (a, c_2) \in G \circ F] \rightarrow c_1 = c_2.$$

Thus  $G \circ F$  is a function.

Now suppose  $a \in \text{dom}(G \circ F)$ . Then we can find a  $c \in \text{ran}(G)$  such that  $(a, c) \in G \circ F$ . This means that there must exist a  $b \in \text{dom}(G)$  such that  $(a, b) \in F \wedge (b, c) \in G$ . But since  $(a, b) \in F$ , we must have  $F(a) = b$  because  $F$  is a function. And since  $(b, c) \in G$  and  $G$  is a function, we must also have  $G(b) = c$ . So  $c = G(b) = G(F(a))$ .

Since  $(a, c) \in G \circ F$  and  $c = G(F(a))$  this means that  $(a, G(F(a))) \in G \circ F$ . And we can also write this as  $(G \circ F)(a) = G(F(a))$  because  $G \circ F$  was proved to be a function.

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Def. An injective function (or injection) is any function  $F$  such that

$$(\forall a_1, a_2 \in \text{dom}(F)) [a_1 \neq a_2 \rightarrow F(a_1) \neq F(a_2)]$$

Ex. 2 Let  $F = \{(1, 5), (2, 6), (3, 7)\}$  &  $G = \{(1, 3), (2, 4), (3, 4)\}$ . Then  $F$  is an injective function but  $G$  is not because  $1 \neq 2$  but  $G(1) = 4 = G(2)$ .

Prop. 2  $F$  is an injective function  $\Leftrightarrow$

$$(\forall a_1, a_2 \in \text{dom}(F)) [F(a_1) = F(a_2) \rightarrow (a_1 = a_2)].$$

Proof:  $F$  is injective  $\Leftrightarrow (\forall a_1, a_2 \in \text{dom}(F)) [a_1 \neq a_2 \rightarrow F(a_1) \neq F(a_2)]$

$$\Leftrightarrow (\forall a_1, a_2 \in \text{dom}(F)) [\neg(a_1 = a_2) \rightarrow \neg\{F(a_1) = F(a_2)\}]$$

$$\Leftrightarrow (\forall a_1, a_2 \in \text{dom}(F)) [\neg\neg\{F(a_1) = F(a_2)\} \rightarrow \neg\neg(a_1 = a_2)]$$

$$\Leftrightarrow (\forall a_1, a_2 \in \text{dom}(F)) [\{F(a_1) = F(a_2)\} \rightarrow (a_1 = a_2)]$$

Prop. 3 If  $F$  is an injective function, then  $F^{-1}$  is also an injective function.

Proof: First observe that since  $F$  is a relation,  $F^{-1}$  is automatically a relation. Also since  $F$  is a function  $F = \{(a, F(a)) : a \in \text{dom}(F)\}$ . So  $F^{-1} = \{(F(a), a) : a \in \text{dom}(F)\}$ . Now suppose  $(b, a_1) \in F^{-1} \wedge (b, a_2) \in F^{-1}$ . Then  $(a_1, b) \in F \wedge (a_2, b) \in F$ . Since  $F$  is a function  $F(a_1) = b \wedge F(a_2) = b$ . So  $F(a_1) = b = F(a_2)$ .  $\therefore F(a_1) = F(a_2)$ . Since  $F$  is an injective function, it follows that  $a_1 = a_2$ .  $\therefore F^{-1}$  is a function. Let us now suppose that  $F^{-1}(b_1) = F^{-1}(b_2) = c$ , say. Then  $(b_1, c) \in F^{-1} \wedge (b_2, c) \in F^{-1}$ . So  $(c, b_1) \in F \wedge (c, b_2) \in F$ . Since  $F$  is a function  $b_1 = b_2$ . So  $F^{-1}(b_1) = F^{-1}(b_2) \rightarrow b_1 = b_2$ .  $\therefore F^{-1}$  is an injective function.

Ex.3

Let  $F = \{(x, \frac{3x}{x-2}) : x \in \mathbb{R} - \{2\}\}$ . Show

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that  $F$  is an injective function.

Sol.

First we will show that  $F$  is a function. Clearly  $x$  is a relation because it is a set of ordered pairs. Note that we had to exclude  $x$  from being 2 because  $\frac{3}{x-2}$  is not defined when  $x=2$ . Now suppose  $(x, y) \in F \wedge (x, z) \in F$ .

Then we must have  $y = \frac{3x}{x-2}$  &  $z = \frac{3x}{x-2}$

so  $y=z$ . So  $(x, y) \in F \wedge (x, z) \in F \rightarrow y=z$ .

$\therefore F$  is a function.

Let us now suppose that  $F(x_1) = F(x_2)$ . Then

$$\frac{3x_1}{x_1-2} = \frac{3x_2}{x_2-2} \text{ so } 3x_1(x_2-2) = 3x_2(x_1-2)$$

$$\therefore 3x_1x_2 - 6x_1 = 3x_1x_2 - 6x_2 \therefore -6x_1 = -6x_2$$

$$\therefore x_1 = x_2 \text{ so } F(x_1) = F(x_2) \rightarrow x_1 = x_2$$

Since this is true for each  $x \in \mathbb{R} - \{2\}$ , it follows that  $F$  is injective. So  $F$  is an injective function.

Prop 4 Let  $F$  and  $G$  be injective functions. Prove that  $G \circ F$  &  $F \circ G$  are also injective functions.

Proof:

Since  $F$  &  $G$  are functions, it follows from Proposition

1 that  $G \circ F$  is a function and  $(G \circ F)(a) =$

$G(F(a))$  for each  $a \in \text{dom}(G \circ F)$ . Now suppose that

$(G \circ F)(a_1) = (G \circ F)(a_2)$ . Then  $G(F(a_1)) = G(F(a_2))$ .

Since  $G$  is injective, it follows that  $F(a_1) = F(a_2)$ .

And since  $F$  is injective, it follows that  $a_1 = a_2$ . So

$G \circ F$  is injective.  $\therefore G \circ F$  is an injective function.

Similarly  $F \circ G$  is an injective function.

## §2. Functions from A to B

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Def. A function from A to B is a 3-tuple  $\langle f, A, B \rangle$ , where f is a function,  $A = \text{dom}(f)$  and B is a set with  $\text{ran}(f) \subseteq B$ .

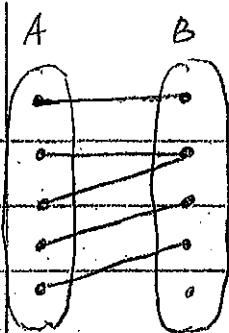
Def. A partial function from A to B is a 3-tuple  $\langle f, A, B \rangle$ , where f is a function and A & B are sets with  $\text{dom}(f) \subseteq A$  &  $\text{ran}(f) \subseteq B$ .

Def. A <sup>total</sup> function on A is just a <sup>total</sup> function from A to A. So it will be a 3-tuple  $\langle f, A, A \rangle$  where f is a function with  $\text{dom}(f) = A$  &  $\text{ran}(f) \subseteq A$ .

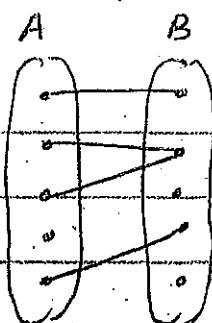
Ex4. Let  $f = \{(1,2), (2,2), (3,3)\}$  &  $g = \{(1,2), (2,3), (4,3)\}$   
 $A = \{1,2,3\}$  and  $B = \{1,2,3,4\}$ . Then

- (a)  $\langle f, A, B \rangle$  is a <sup>total</sup> function from A to B.
- (b)  $\langle g, A, B \rangle$  is not a function from A to B b.c.  $\text{dom}(g) \neq A$ .
- (c)  $\langle f, A, A \rangle$  is a <sup>total</sup> function on A.
- (d)  $\langle g, B, B \rangle$  is not a <sup>total</sup> function on B b.c.  $\text{dom}(g) \neq B$ .
- (e)  $\langle g, B, A \rangle$  is a partial function from B to A.
- (f)  $\langle g, B, B \rangle$  is a partial function on B.
- (g)  $\langle g, A, B \rangle$  is not a partial function from A to B because  $\text{dom}(g) \not\subseteq A$ .

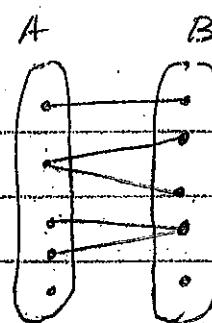
Notation Instead of writing  $\langle f, A, B \rangle$  is a <sup>total</sup> function from A to B we usually write  $f: A \rightarrow B$  is a function from A to B. We call A the domain of  $f: A \rightarrow B$  because  $\text{dom}(f) = A$  and we call B the codomain of  $f: A \rightarrow B$ . If  $f: A \rightarrow B$  is a partial function from A to B, we usually say that A is the source space & B is the target space.



1. A function from A to B  
2. A partial function from A to B



1. Not a func. from A to B  
2. A partial function from A to B



1. Not a func. from A to B  
2. Not a partial function from A to B

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Def. Let  $f: A \rightarrow B$  (i.e.,  $\langle f, A, B \rangle$ ) be a function from A to B. We say that  $f: A \rightarrow B$  is a surjective function from A to B (or surjection from A to B) if  $(\forall b \in B)(\exists a \in A)[f(a) = b]$ .

that

Prop. 5  $f: A \rightarrow B$  is surjective  $\Leftrightarrow \text{ran}(f) = B$

Proof: First observe that for  $f$  to be a function from A to B we always need to have  $\text{ran}(f) \subseteq B$ . Now  $f: A \rightarrow B$  is surjective  $\Leftrightarrow (\forall b \in B)(\exists a \in A)[f(a) = b] \quad \& \quad \text{ran}(f) \subseteq B$   
 $\Leftrightarrow (\forall b \in B)(\exists a \in A)[(a, b) \in f] \quad \& \quad \text{ran}(f) \subseteq B$   
 $\Leftrightarrow (\forall b \in B)[b \in \text{ran}(f)] \quad \& \quad \text{ran}(f) \subseteq B$   
 $\Leftrightarrow B \subseteq \text{ran}(f) \quad \& \quad \text{ran}(f) \subseteq B$   
 $\Leftrightarrow \text{ran}(f) = B$ .

Ex. 5 Let  $A = \mathbb{R} - \{2\}$  &  $B = \mathbb{R} - \{3\}$ , let  $f: A \rightarrow B$  be the function from A to B with  $f = \{(x, \frac{3x}{x-2}) : x \in \mathbb{R} - \{2\}\}$ . Show that  $f: A \rightarrow B$  is surjective.

Sol. Let  $y$  be any element of  $\mathbb{R} - \{3\}$ . We must show that there is an  $x \in \mathbb{R} - \{2\}$  such that  $f(x) = y$ . So we want an  $x$  such that  $\frac{3x}{x-2} = y$ . Now

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Ex.5

$$\begin{aligned}
 \frac{3x}{x-2} = y &\Leftrightarrow 3x = y(x-2) \\
 &\Leftrightarrow 3x = yx - 2y \\
 &\Leftrightarrow 2y = yx - 3x \\
 &\Leftrightarrow 2y = x(y-3) \Leftrightarrow x = \frac{2y}{y-3}.
 \end{aligned}$$

So if we take  $x = 2y/(y-3)$  we will get  $f(x) = y$ .

Now this  $x$  is defined for any  $y \neq 3$ , so  $f(x) = y$  for each  $y \in \mathbb{R} - \{3\}$ . Also

$$x = \frac{2y}{y-3} = \frac{2(y-3)+6}{(y-3)} = 2 + \frac{6}{y-3}.$$

Since  $\frac{6}{y-3}$  can never be zero for any  $y \in \mathbb{R} - \{3\}$ , it follows that  $x$  can never be 2. So  $x \in \mathbb{R} - \{2\}$ .

Hence  $(\forall y \in B)(\exists x \in A)[f(x) = y]$ . So  $f: A \rightarrow B$  is surjective.

Def. Let  $f: A \rightarrow B$  be a function from  $A$  to  $B$ . We say that  $f: A \rightarrow B$  is a bijective function from  $A$  to  $B$  (or a bijection from  $A$  to  $B$ ) if  $f: A \rightarrow B$  is both injective and surjective.

Ex.6 Let  $f: \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{3\}$  be the function from  $\mathbb{R} - \{2\}$  to  $\mathbb{R} - \{3\}$  defined by  $f(x) = 3x/(x-2)$ . Then in Ex.3 we verified that  $f$  was injective and in Ex.5 we verified that  $f: \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{3\}$  was surjective. So  $f: \mathbb{R} - \{2\} \rightarrow \mathbb{R} - \{3\}$  is bijective.

Note: We can say that a function  $f$  is injective without any reference to a codomain. But we can only talk about surjective functions from  $A$  to  $B$  because without the codomain  $B$ , the concept of surjective is meaningless.

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Prop 6

Suppose  $f: A \rightarrow B$  &  $g: B \rightarrow C$  are surjective.  
 Then  $gof: A \rightarrow C$  is also surjective.

Proof:

Suppose  $f: A \rightarrow B$  &  $g: B \rightarrow C$  are surjective.  
 In order to show that  $gof: A \rightarrow C$  is surjective,  
 we must show that  $(\forall c \in C)(\exists a \in A)[(gof)(a) = c]$

So let  $c \in C$  be an arbitrary element of  $C$ . Since  
 $g: B \rightarrow C$  is surjective, we can find a  $b \in B$   
 such that  $g(b) = c$ . Also since  $f: A \rightarrow B$  is  
 surjective, we can find an  $a \in A$  such that  
 $f(a) = b$  for this same  $b$ . So  $g(f(a)) = g(b) = c$ .  
 Hence  $(gof)(a) = c$ . Since  $c$  was arbitrary,  
 it follows that  $gof$  is surjective.

Note: If  $f: A \rightarrow B$  is a bijective function from  $A$  to  $B$ ,  
 then  $f^{-1}: B \rightarrow A$  is also a bijective function from  $B$  to  $A$ .

Proof:

Do for H.W.

Prop 7

Let  $f: A \rightarrow B$  be a bijective function from  $A$  to  $B$ .Then (a)  $f^{-1} \circ f = i_A$       (b)  $f \circ f^{-1} = i_B$ .

Proof:

Recall that  $i_A = \{(a, a) : a \in A\}$ . So  $i_A$  is a function  
 from  $A$  to  $A$ . It is called the identity function on  $A$ .

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Now let  $a \in A$  be an arbitrary element of  $A$ . Put  
 $b = f(a)$ . Then  $b \in B$  and  $(a, b) \in f$ . So  $(b, a) \in f^{-1}$ .  
 Thus  $f^{-1}(b) = a$ . Hence  $(f^{-1} \circ f)(a) = f^{-1}(f(a))$  by Prop. 1  
 $= f^{-1}(b) = a$ . So  $f^{-1} \circ f = \{(a, a) : a \in A\} = i_A$ .

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- (b) Also let  $b \in B$  be an arbitrary element of  $B$ . Put  
 $a = f^{-1}(b)$ . Then  $a \in A$  and  $(b, a) \in f'$ . So  
 $(a, b) \in f$ . Hence  $f(a) = b$ . Thus

$$\begin{aligned} (f \circ f^{-1})(b) &= f(f^{-1}(b)) \quad \text{by Prop. 1} \\ &= f(a) \quad \text{bec. } a = f^{-1}(b) \\ &= b \quad \text{bec. } f(a) = b. \end{aligned}$$

Since  $b$  was arbitrary it follows that

$$f \circ f^{-1} = \{(b, b) : b \in B\} = i_B.$$

- Prop.8 Let  $f: A \rightarrow B$  be a function from  $A$  to  $B$ . If there is a function  $g: B \rightarrow A$  such that  $g \circ f = i_A$ , then  $f$  is injective.  
 (a) function  $g: B \rightarrow A$  such that  $f \circ g = i_B$ , then  $f$  is surjective.

Proof: (a) Suppose  $a_1, a_2 \in A$  and  $f(a_1) = f(a_2)$ . Then

$$g(f(a_1)) = g(f(a_2)) \quad \text{bec. } f(a_1) = f(a_2).$$

$$\text{So } (g \circ f)(a_1) = (g \circ f)(a_2)$$

$$\therefore i_A(a_1) = i_A(a_2) \quad \text{bec. } g \circ f = i_A$$

$$\therefore a_1 = a_2. \quad \text{So } f(a_1) = f(a_2) \rightarrow (a_1 = a_2).$$

Since  $a_1$  &  $a_2$  were arbitrary, it follows that  $f$  is injective.

- (b) Let  $b \in B$  be an arbitrary element of  $B$ . Put  
 $a = g(b)$ . Then  $a \in A$ . Also

$$\begin{aligned} f(a) &= f(g(b)) \quad \text{bec. } a = g(b) \\ &= (f \circ g)(b) \\ &= i_B(b) = b. \end{aligned}$$

Since  $b \in B$  was arbitrary, it follows that  
 $(\forall b \in B)(\exists a \in A)[f(a) = b]$ . So  $f: A \rightarrow B$  is a surjective.

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Theorem 9: Suppose  $f: A \rightarrow B$  is a function from  $A$  to  $B$  and  $g: B \rightarrow A$  is a function from  $B$  to  $A$ . Suppose also that  $g \circ f = i_A$  and  $f \circ g = i_B$ . Then  $g = f^{-1}$ .

Proof: First observe that since  $g \circ f = i_A$ , it follows from Prop. 8(a) that  $f: A \rightarrow B$  is injective. Also since  $f \circ g = i_B$ , it follows from Prop. 8(b) that  $f: A \rightarrow B$  is surjective. So  $f: A \rightarrow B$  is bijective. Hence  $f^{-1}: B \rightarrow A$  is a function.

$$\begin{aligned}
 \text{Thus } g &= i_A \circ g \\
 &= (f^{-1} \circ f) \circ g \quad \text{bec. } f \circ f^{-1} = i_A \\
 &= f^{-1} \circ (f \circ g) \quad \text{bec. "o" is associative} \\
 &= f^{-1} \circ i_B \\
 &= f^{-1}.
 \end{aligned}$$

Hence  $g = f^{-1}$  and we are done.

Ex. 7 Let  $f: R - \{3\} \rightarrow R - \{1\}$  be the function from  $R - \{3\}$  to  $R - \{1\}$  with  $f(x) = (x+2)/(x-3)$ . Find  $f^{-1}(x)$ .

Sol. For each  $x \in R - \{1\}$ , let  $y = f^{-1}(x)$ . Then

$$f(y) = f(f^{-1}(x)) = (f \circ f^{-1})(x) = x.$$

Since  $f(x) = \frac{x+2}{x-3}$ ,  $f(y) = \frac{y+2}{y-3}$ . So

$$\frac{y+2}{y-3} = x. \quad \therefore y+2 = x(y-3)$$

$$\text{Hence } 2+3x = xy - y. \quad \therefore 3x+2 = y(x-1)$$

$$\therefore y = (3x+2)/(x-1). \quad \text{So } f^{-1}(x) = y = \frac{3x+2}{x-1}$$

for each  $x \in R - \{1\}$ .

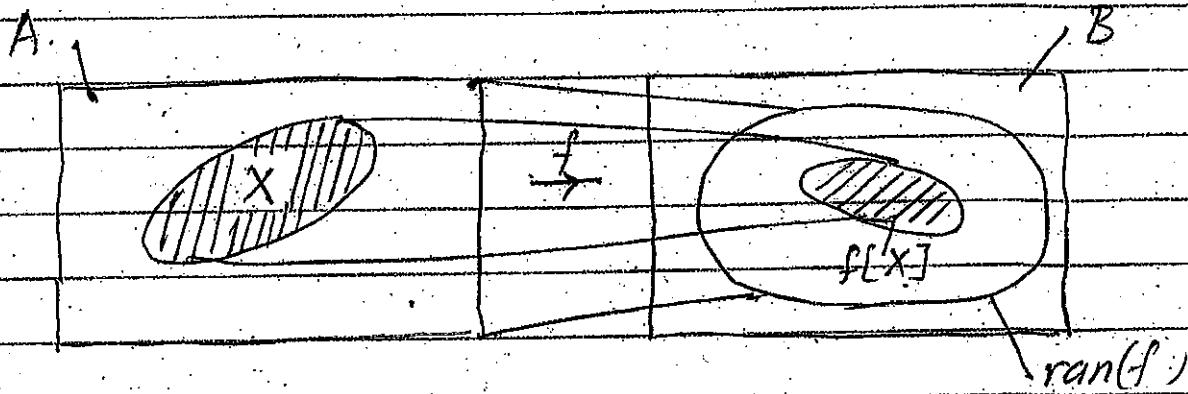
### §3 Images & Pre-images of a function from A to B

Def Let  $f: A \rightarrow B$  be a function from  $A$  to  $B$ , and  $X \subseteq A$ . We define the image of  $X$  under  $f$  by  $f[X] = \{f(a) : a \in X\}$ . Note that  $f[X] \subseteq B$  and that  $f[A] = \{f(a) : a \in A\} = \text{ran}(f)$ . Note that we can also write  $f[X]$  as  $\{b \in B : (\exists a \in X)(b = f(a))\}$ .

Ex 1 Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by putting  $f(a) = a^2$ , and  $X = \{-2, -1, 0, 1, 2\}$ . Then

$$\begin{aligned} f[X] &= \{f(a) : a \in X\} = \{f(-2), f(-1), f(0), f(1), f(2)\} \\ &= \{4, 1, 0, 1, 4\} = \{0, 1, 4\}. \end{aligned}$$

Also  $f[\mathbb{Z}] = \{f(a) : a \in \mathbb{Z}\} = \{a^2 : a \in \mathbb{Z}\} = \{a^2 : a \in \mathbb{N}\}$  and  $f[\emptyset] = \{f(a) : a \in \emptyset\} = \{\} = \emptyset$ .



Theorem 10: Let  $f: A \rightarrow B$  be a function from  $A$  to  $B$  and  $W \subseteq A$  &  $X \subseteq A$  be subsets of  $A$ . Then

- (a)  $f[W \cup X] = f[W] \cup f[X]$
- (b)  $f[W \cap X] \subseteq f[W] \cap f[X]$  but  $f[W] \cap f[X] \neq f[W \cap X]$  in general.
- (c)  $f[W] - f[X] \subseteq f[W - X]$  but  $f[W - X] \neq f[W] - f[X]$  in general
- (d)  $W \subseteq X \Rightarrow f[W] \subseteq f[X]$  but  $f[W] \subseteq f[X] \not\Rightarrow W \subseteq X$  in general

Proof: (a) Let  $b \in f[W \cup X]$ . Then we can find an  $a \in W \cup X$  such that  $b = f(a)$ . Since  $a \in W \cup X$ , then  $a \in W$  or  $a \in X$ . Now if  $a \in W$ , then  $b = f(a) \in f[W]$ ; and if  $a \in X$ , then  $b = f(a) \in f[X]$ . So  $b \in f[W]$  or  $b \in f[X]$ . Hence  $b \in f[W] \cup f[X]$ . Thus  $f[W \cup X] \subseteq f[W] \cup f[X] \dots (1)$

Now suppose  $b \in f[W] \cup f[X]$ . Then  $b \in f[W]$  or  $b \in f[X]$ . Now if  $b \in f[W]$ , then we can find an  $a_1 \in W$  such that  $b = f(a_1)$ . Since  $a_1 \in W$ ,  $a_1 \in W \cup X$ . So  $b = f(a_1) \in f[W \cup X]$ . And if  $b \in f[X]$ , then we can find an  $a_2 \in X$  such that  $b = f(a_2)$ . Since  $a_2 \in X$ ,  $a_2 \in W \cup X$ . So  $b = f(a_2) \in f[W \cup X]$ . Hence in either case  $b \in f[W \cup X]$ . Thus  $f[W] \cup f[X] \subseteq f[W \cup X] \dots (2)$ .

From (1) & (2), we get  $f[W \cup X] = f[W] \cup f[X]$ .

(b) Suppose  $b \in f[W \cap X]$ . Then we can find an  $a \in W \cap X$  such that  $b = f(a)$ . Since  $a \in W \cap X$ , then  $a \in W$  and  $a \in X$ . So  $b = f(a) \in f[W]$  because  $a \in W$  and  $b = f(a) \in f[X]$  because  $a \in X$ . Hence  $b \in f[W] \cap f[X]$ . Thus  $f[W \cap X] \subseteq f[W] \cap f[X]$ .

Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(a) = a^2$  and put  $W = \{-2, -1, 0\}$  and  $X = \{0, 1, 3\}$ . Then  $W \cap X = \{0\}$ .

$$\text{So } f[W \cap X] = \{f(0)\} = \{0\}$$

$$f[W] = \{f(-2), f(-1), f(0)\} = \{0, 1, 4\}$$

$$f[X] = \{f(0), f(1), f(3)\} = \{0, 1, 9\}$$

Thus  $f[W] \cap f[X] = \{0, 1\} \neq \{0\} = f[W \cap X]$ . Hence  $f[W] \cap f[X] \not\subseteq f[W \cap X]$  in general.

(c) Suppose  $b \in f[W] - f[X]$ . Then  $b \in f[W]$  and (4)  
 $b \notin f[X]$ . So we can find an  $a \in W$  such that  
 $b = f(a)$ . Also if  $a \in X$ , then we would have  
 $b = f(a) \in f[X]$ . Since  $b \notin f[X]$ , it follows that  
 $a \notin X$ . So  $a \in W$  and  $a \notin X$ . Hence  $a \in W - X$ .  
 Thus  $b = f(a) \in f[W-X]$ . So  $f[W] - f[X] \subseteq f[W-X]$ .

Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(a) = a^2$  and  
 put  $W = \{-2, 0\}$  and  $X = \{0, 2\}$ . Then  $W-X = \{-2\}$ .  
 So  $f[W-X] = \{f(-2)\} = \{4\}$ . Also  
 $f[W] = \{f(-2), f(0)\} = \{0, 4\}$   
&  $f[X] = \{f(0), f(2)\} = \{0, 4\}$ . Thus  
 $f[W-X] = \{4\} \neq \emptyset = f[W] - f[X]$ .  
 Hence  $f[W-X] \not\subseteq f[W] - f[X]$  in general.

(d) Assume that  $W \subseteq X$ . Then  $a \in W \Rightarrow a \in X$ . Now  
 suppose that  $b \in f[W]$ . Then we can find  
 an  $a \in W$  such that  $b = f(a)$ . Since  $a \in W$ ,  $a \in X$ .  
 So  $b = f(a) \in f[X]$ . Hence  $f[W] \subseteq f[X]$ . So  
 $W \subseteq X \Rightarrow f[W] \subseteq f[X]$ .

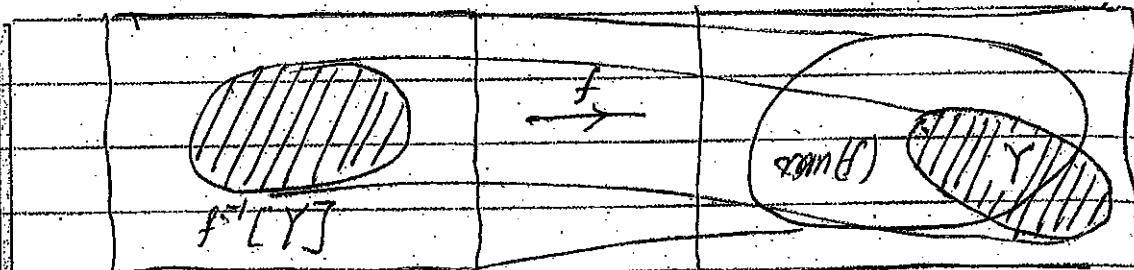
Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(a) = a^2$  and  
 put  $W = \{-2, 0\}$  and  $X = \{0, 1, 2\}$ . Then  $W \not\subseteq X$ .  
 Also  $f[W] = \{f(-2), f(0)\} = \{0, 4\}$  and  
 $f[X] = \{f(0), f(1), f(2)\} = \{0, 1, 4\}$ .  
 So  $f[W] = \{0, 4\} \subseteq \{0, 1, 4\} = f[X]$  but  $W \not\subseteq X$ .  
 Hence  $f[W] \subseteq X \not\Rightarrow W \subseteq X$  in general.

Def. Let  $f: A \rightarrow B$  be a function from  $A$  to  $B$  and  $Y \subseteq B$ . We define the pre-image (or inverse image) of  $Y$  under  $f$  by

$$f^{-1}[Y] = \{a \in A : f(a) \in Y\}$$

Note that  $f^{-1}[Y] \subseteq A$ . Also  $f^{-1}[B] = f^{-1}[\text{ran}(f)] = A = \text{dom}(f)$ . We can also write  $f^{-1}[Y]$  as  $f^{-1}[Y] = \{a \in A : (\exists b \in Y)(f(a) = b)\}$ .

And note that  $a \in f^{-1}[Y] \Leftrightarrow f(a) \in Y$ .



Ex. 2 Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(a) = a^2$ . Put  $Y = \{-3, -1, 0, 1, 4\}$ . Then

$$f^{-1}[Y] = \{-2, -1, 0, 1, 2\} \text{ because } f(-2) = 4 \in Y, f(-1) = 1 \in Y, f(0) = 0 \in Y, f(1) = 1 \in Y.$$

Note that  $f^{-1}[\{-3, -1\}] = \emptyset$  and in general  $f^{-1}[Y] = f^{-1}[Y \cap \text{ran}(f)]$ .

Theorem 17: Let  $f: A \rightarrow B$  be a function from  $A$  to  $B$  and  $Y \subseteq B$  &  $Z \subseteq B$ . Then

$$(a) f^{-1}[Y \cup Z] = f^{-1}[Y] \cup f^{-1}[Z]$$

$$(b) f^{-1}[Y \cap Z] = f^{-1}[Y] \cap f^{-1}[Z]$$

$$(c) f^{-1}[Y - Z] = f^{-1}[Y] - f^{-1}[Z]$$

$$(d) Y \subseteq Z \Rightarrow f^{-1}[Y] \subseteq f^{-1}[Z] \text{ but } f^{-1}[Y] \subseteq f^{-1}[Z] \not\Rightarrow Y \subseteq Z \text{ in general.}$$

Proof: (a) Do for Homework

(b) Let  $a \in f^{-1}[Y \cap Z]$ . Then  $f(a) \in Y \cap Z$ . So  $f(a) \in Y$  and  $f(a) \in Z$ . Thus  $a \in f^{-1}[Y]$  and  $a \in f^{-1}[Z]$ . Hence  $a \in f^{-1}[Y] \cap f^{-1}[Z]$ . Thus  $f^{-1}[Y \cap Z] \subseteq f^{-1}[Y] \cap f^{-1}[Z]$ . (1)  
 Now suppose  $a \in f^{-1}[Y] \cap f^{-1}[Z]$ . Then  $a \in f^{-1}[Y]$  and  $a \in f^{-1}[Z]$ . So  $f(a) \in Y$  and  $f(a) \in Z$ . Thus  $f(a) \in Y \cap Z$ . Hence  $a \in f^{-1}[Y \cap Z]$ . Thus  
 $f^{-1}[Y] \cap f^{-1}[Z] \subseteq f^{-1}[Y \cap Z] \dots (2)$ . From (1)  
& (2), it follows that  $f^{-1}[Y \cap Z] = f^{-1}[Y] \cap f^{-1}[Z]$ .

(c) Suppose  $a \in f^{-1}[Y - Z]$ . Then  $f(a) \in Y - Z$ . So  $f(a) \in Y$  and  $f(a) \notin Z$ . Thus  $a \in f^{-1}[Y]$  and  $a \notin f^{-1}[Z]$  because  $a \in f^{-1}[Z] \Leftrightarrow f(a) \in Z$ . Hence  $a \in f^{-1}[Y] - f^{-1}[Z]$ . So  $f^{-1}[Y - Z] \subseteq f^{-1}[Y] - f^{-1}[Z] \dots (1)$   
 Now suppose  $a \in f^{-1}[Y] - f^{-1}[Z]$ . Then  $a \in f^{-1}[Y]$  and  $a \notin f^{-1}[Z]$ . So  $f(a) \in Y$  and  $f(a) \notin Z$ . Hence  $f(a) \in Y - Z$ . Thus  $a \in f^{-1}[Y - Z]$ . So  
 $f^{-1}[Y] - f^{-1}[Z] \subseteq f^{-1}[Y - Z] \dots (2)$ . From (1) & (2), it follows that  $f^{-1}[Y - Z] = f^{-1}[Y] - f^{-1}[Z]$ .

(d) Assume that  $Y \subseteq Z$ . Let  $a \in f^{-1}[Y]$ . Then  $f(a) \in Y$ . Since  $Y \subseteq Z$ , we get that  $f(a) \in Z$ . Thus  $a \in f^{-1}[Z]$ .  
 $\therefore f^{-1}[Y] \subseteq f^{-1}[Z]$ . Hence  $Y \subseteq Z \Rightarrow f^{-1}[Y] \subseteq f^{-1}[Z]$ .

Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(a) = a^2$  &  $Y = \{-2, 1\}$  and  $Z = \{0, 1\}$ . Then  $Y \not\subseteq Z$  and  $f^{-1}[Y] = \{-1, 1\}$  &  $f^{-1}[Z] = \{-1, 0, 1\}$ . So  $f^{-1}[Y] = \{-1, 1\} \subseteq \{-1, 0, 1\} = f^{-1}[Z]$  but  $Y \not\subseteq Z$ . Hence  $f^{-1}[Y] \subseteq f^{-1}[Z] \nRightarrow Y \subseteq Z$  in general.

(17)

Prop. 12 Let  $f: A \rightarrow B$  be a function from  $A$  to  $B$  and  
 $X \subseteq A$  &  $Y \subseteq B$ . Then

- (a)  $X \subseteq f^{-1}[f[X]]$  but  $f^{-1}[f[X]] \not\subseteq X$  in general
- (b)  $f[f^{-1}[Y]] \subseteq Y$  but  $Y \not\subseteq f[f^{-1}[Y]]$  in general.

Proof. (a) Suppose  $a \in X$ . Then  $f(a) \in f[X]$ . So  $a \in f^{-1}[f[X]]$  because  $f(a) \in f[X]$ . Thus  $X \subseteq f^{-1}[f[X]]$

Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(a) = a^2$  and  $X = \{0, 2\}$

Then  $f[X] = \{f(0), f(2)\} = \{0, 4\}$ . So

$$f^{-1}[f[X]] = f^{-1}[\{0, 4\}] = \{-2, 0, 2\} \not\subseteq \{0, 2\} = X.$$

Thus  $f^{-1}[f[X]] \not\subseteq X$  in general.

(b) Suppose  $b \in f[f^{-1}[Y]]$ . Then we can find an  $a \in f^{-1}[Y]$  such that  $b = f(a)$ . But  $a \in f^{-1}[Y] \iff f(a) \in Y$ . So  $f(a) \in Y$ . Since  $b = f(a)$ , it follows that  $b \in Y$ . Hence  $f[f^{-1}[Y]] \subseteq Y$ .

Let  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  be defined by  $f(a) = a^2$ . Put

$Y = \{-1, 0, 4\}$ . Then  $f[Y] = \{0, -2, 2\}$ . So

$$f[f^{-1}[Y]] = \{f(0), f(-2), f(2)\} = \{0, 4\}; \text{ So}$$

$Y = \{-1, 0, 4\} \not\subseteq \{0, 4\} = f[f^{-1}[Y]]$ . Hence  
 $Y \not\subseteq f[f^{-1}[Y]]$  in general.

Note: 1. If  $f: A \rightarrow B$  is injective and  $W, X \subseteq A$ , then

$$(a) f[W \cap X] = f[W] \cap f[X] \text{ and}$$

$$(b) f[W - X] = f[W] - f[X].$$

2. If  $f: A \rightarrow B$  is surjective, then  $Y \subseteq B \iff f^{-1}[Y] \subseteq f[Z]$ .

## §4. Finite sequences, $n$ -ary relations, & $n$ -ary functions

Recall that an ordered pair  $(a, b)$  was defined by  $(a, b) = \{\{a\}, \{a, b\}\}$ . Recall also that an indexed family of sets  $\langle A_i : i \in I \rangle$  was just a function  $g : I \rightarrow V$ , where  $A_i = g(i)$ .

Finally, recall that the Cartesian product of  $A$  followed by  $B$  was defined by

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

Def. Let  $\langle A_i : i \in I \rangle$  be an indexed family of sets. We define the Generalized Cartesian product of the indexed family by

$$\prod_{i \in I} A_i = \{f : f \text{ is a function with } \text{dom}(f) = I \text{ and } f(i) \in A_i \text{ for each } i \in I\}$$

Ex. 1 Let  $I = \{1, 2, 3\}$  and  $A_1 = \{a, b\}$ ,  $A_2 = \{c, d\}$ , &  $A_3 = \{e\}$ . Then

$$\prod_{i \in I} A_i = \{f_1, f_2, f_3, f_4\} \quad \text{where}$$

$$f_1 = \{(1, a), (2, c), (3, e)\}, \quad f_2 = \{(1, a), (2, d), (3, e)\}$$

$$f_3 = \{(1, b), (2, c), (3, e)\}, \quad \& \quad f_4 = \{(1, b), (2, d), (3, e)\}.$$

Notation: If  $\langle A_i : i \in I \rangle$  is an indexed family of sets with  $A_i = A$  for each  $i \in I$ , then we usually write  $\prod_{i \in I} A_i = \prod_{i \in I} A$  as  $A^I$ . So

$$A^I = \{f : f \text{ is a function with } \text{dom}(f) = I \text{ & } \text{ran}(f) \subseteq A\}$$

Some writers use the notation  $\mathcal{F}(I, A)$  to denote  $A^I$ .

Def. For each  $n \in \mathbb{N}$ , define  $\mathbb{N}_n$  by  $\mathbb{N}_0 = \emptyset$  and  $\mathbb{N}_n = \{0, 1, 2, \dots, n-1\}$  if  $n \geq 1$ . A finite sequence is just a function  $f$  with  $\text{dom}(f) = \mathbb{N}_n$ .

(19)

Ex. 2 Let  $f = \{(0, b), (1, a), (2, c)\}$ . Then  $f$  is a sequence. The empty set of ordered pairs is  $\emptyset$  - and it satisfies the condition for being a function. In order to emphasize that we are thinking of  $\emptyset$  as a function we will use the notation  $\langle \rangle$  to denote it - and call it the empty function.

Notation: Since a finite sequence always has  $\mathbb{N}_n$  as its domain, we only need to specify  $f(0), \dots, f(n-1)$  to identify the finite sequence. So we usually write the function  $f = \{(0, b), (1, a), (2, c)\}$  as  $\langle b, a, c \rangle$  with pointed brackets. When the finite sequence is written this way, we usually call it an  $n$ -tuple (and in this case  $n=3$ ).

Def. An  $n$ -tuple is a finite sequence  $f$  with domain  $\mathbb{N}_n$  and it is usually written as  $\langle f(0), f(1), \dots, f(n-1) \rangle$ .

Def. We define the set of all finite sequences of  $n$  elements of the set  $A$  by

$$A^n = \{f : f \text{ is an } n\text{-tuple with } \text{dom}(f) \subseteq A\}$$

We also define the set of all finite sequences of elements of  $A$  by  $A^* = \bigcup_{n \in \mathbb{N}} A^n$

Ex. 3 Let  $A = \{a, b\}$ . Then  $A^0 = \{\langle \rangle\}$ ,  $A' = \{\langle a \rangle, \langle b \rangle\}$  (20) and  $A^2 = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, a \rangle, \langle b, b \rangle\}$ . Also  $A^* = \{\langle \rangle, \langle a \rangle, \langle b \rangle, \langle a, a \rangle, \langle a, b \rangle, \dots, \langle a, a, a \rangle, \dots\}$ . Note:  $\langle b, a \rangle$  is an abbreviation of the function  $f = \{(0, b), (1, a)\}$  and it is called a 2-tuple.

Def. An  $n$ -ary relation  $R$  on the set  $A$  is just a subset of  $A^n$ . An  $n$ -ary function  $f$  from  $A$  to  $B$  is just a function  $f$  from  $A^n$  to  $B$ . An  $n$ -ary  $m$ -vector-valued function from  $A$  to  $B$  is just a function from  $A^n$  to  $B^m$ .

Ex. 4(a) Let  $A = \{a, b\}$ . Put  $R = \langle a, b, b \rangle, \langle b, a, b \rangle, \langle b, b, a \rangle \}$ . Then  $\langle R, A^3 \rangle$  is a ternary (3-ary) relation on  $A$ .  $\langle \emptyset, A^3 \rangle$  &  $\langle \emptyset, A \rangle$  are ternary & unary (1-ary) relations... on  $A$ .

(b) Let  $A = \{a, b\}$  and  $B = \{1, 2, 3, 4\}$ . Put

$$f = \{\langle \langle a, a \rangle, 1 \rangle, \langle \langle a, b \rangle, 2 \rangle, \langle \langle b, a \rangle, 2 \rangle, \langle \langle b, b \rangle, 3 \rangle\}$$

Then  $\langle f, A^2, B \rangle$  is a binary (2-ary) function from  $A$  to  $B$ .

(c) Let  $A = \{a, b\}$  and  $B = \{1, 2, 3\}$ . Put

$$f = \{\langle \langle a, a \rangle, \langle 1, 1 \rangle \rangle, \langle \langle a, b \rangle, \langle 1, 2 \rangle \rangle, \langle \langle b, a \rangle, \langle 2, 1 \rangle \rangle, \langle \langle b, b \rangle, \langle 2, 2 \rangle \rangle\}$$

Then  $\langle f, A^2, B^2 \rangle$  is a binary 2-vector-valued function from  $A$  to  $B$ .

Note A 0-ary (nullary) relation  $R$  on  $A$  is a subset of  $A^0$ . So  $\langle \emptyset, A^0 \rangle$  &  $\langle A^0, A^0 \rangle$  are the only 2 0-ary relations on  $A$ .

A 0-ary (nullary) function  $f$  on the set  $A$  is just a function from  $A^0 = \{\langle \rangle\}$  to  $A$ . So it is like a constant from  $A$ .

(21)

Ex.5 Let  $f: \mathbb{R}^{\vee} \rightarrow \mathbb{R}$  be defined by putting  $f(\langle \rangle) = \pi$ . Then  $f = \{\langle \rangle, \pi\}$  but we can think of  $f$  as just the constant  $\pi$  (which is  $3.1415926\dots$ )

Ex.6 Let us show what is  $A^*$  for various  $A$ 's.

- (a) If  $A = \{a\}$ , then  $A^* = \{\langle \rangle, \langle a \rangle, \langle a, a \rangle, \langle a, a, a \rangle, \dots\}$
- (b) If  $A = \{0, 1\}$ , then  $A^* = \{\langle \rangle, \langle 0 \rangle, \langle 1 \rangle, \langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \dots\}$
- (c) If  $A = \emptyset$ , then  $A^* = \{\langle \rangle\}$ .

Def. A multiset  $M$  is a 2-tuple  $\langle A, f \rangle$  where  $A$  is a set and  $f$  is a function with  $\text{dom}(f) = A$  and  $\text{ran}(f) \subseteq \mathbb{Z}^+ \cup \{00\} = \{00, 1, 2, 3, \dots\}$ .  $A$  is called the underlying set of the multiset. (Some authors say that a multiset is just a function  $f$  with  $\text{ran}(f) \subseteq \mathbb{Z}^+ \cup \{00\}$ .)

Ex.7 Let  $A = \{a, b\}$  and put  $f = \{(a, 3), (b, 2)\}$ . Then  $M = \langle A, f \rangle$  is a multiset. We usually write  $M$  as  $[3.a, 2.b]$  but we can also write  $M$  as  $[a, a, a, b, b]$  or even as  $[a, b, b, a, a]$ .

Note: We have encountered various kinds of objects which are denoted by using various kinds of brackets. Some of these objects are also empty objects – but they are all different (in some sense).

- (a) Empty set,  $\emptyset = \{\}$ ;  $\langle a, b \rangle = \{(0, a), (1, b)\}$
- (b) Empty sequence (0-tuple)  $\langle \rangle = \{\}$ ;  $(a, b) = \{[a], [a, b]\}$
- (c) Empty multiset,  $[\ ] = \langle \emptyset, \langle \rangle \rangle$
- (d) Singleton set  $\{a\}$ , singleton multiset  $[a]$ , 1-tuple  $\langle a \rangle$
- (e) pair  $\{a, b\}$ ,  $a \neq b$ ; ordered pair  $(a, a)$ , 2-tuple  $\langle a, a \rangle$ ; multiset  $[a, a]$ .

Now that we have encountered the notions of a multiset and of a finite sequence, we naturally would like to discuss sub-multisets and finite subsequences. (22)

Ex. 8 Let  $M = [3.a, 2.b, 4.c]$ . Then  $[2.a, 1.b]$  and  $[2.b, 3.c]$  are both sub-multisets of  $M$  but  $[2.a, 3.b]$  is not a sub-multiset of  $M$ . Also if  $\underline{s} = \langle a_1, a_2, a_3, a_4 \rangle$ , then  $\langle a_1, a_3, a_4 \rangle$  and  $\langle a_2, a_4 \rangle$  are both subsequences of  $\underline{s}$  but  $\langle a_2, a_1, a_3 \rangle$  &  $\langle a_2, a_2, a_3 \rangle$  are not subsequence of  $\underline{s}$ .

Def. The multiset  $\langle A, f \rangle$  is a sub-multiset of the multiset  $\langle B, g \rangle$  if  $A \subseteq B$  and  $f(a) \leq g(a)$  for each  $a \in A$ .

Def. The function  $g: \mathbb{N}_k \rightarrow \mathbb{N}_n$  is strictly increasing if for each  $i, j \in \mathbb{N}_k$ ,  $i < j \Rightarrow g(i) < g(j)$ .

Ex. 9 Let  $g: \{0, 1, 2, \dots, 4\} \rightarrow \{0, 1, 2, \dots, 16\}$  be defined by  $g(i) = 3i + 1$ . Then  $g$  is a strictly increasing function from  $\mathbb{N}_5$  to  $\mathbb{N}_{17}$  &  $g = \{(0, 1), (1, 4), (2, 7), (3, 10), (4, 13)\}$ .

Def. A subsequence of the sequence  $f$  is an ordered pair of the form  $(f, g)$  where  $g$  is a strictly increasing function from  $\mathbb{N}_k$  to  $\mathbb{N}_n = \text{dom}(f)$ .

Ex. 10 Let  $g$  be as in Ex. 9 &  $f = \{(0, a_0), (1, a_1), \dots, (16, a_{16})\}$ . Then  $(f, g) = \{( (0, a_0), (1, a_1), \dots, (16, a_{16}) \}, \{ (0, 1), (1, 4), (2, 7), (3, 10), (4, 13) \} \}$  is a subsequence of  $f$ . We usually write  $(f, g)$  as the ordered pair of "sequences"  $(\underbrace{\langle a_0, a_1, a_2, \dots, a_{15}, a_{16} \rangle}_{f}, \underbrace{\langle 1, 4, 7, 10, 13 \rangle}_{g})$