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Ch. 6 - The Cardinality of Set

§1. Equinumerous sets and finite sets.

A fundamental question in mathematics is to define what it means for a set to be finite. We could say that a finite set is one that is not infinite. But what does it mean for a set to be infinite? To answer this question, we introduce the concept of equinumerous — which basically tells us when two sets have the same size.

Def. Let A & B be sets. We define A to be equinumerous with B if there exists a bijection $f: A \rightarrow B$. We will write $A \approx B$ to indicate that A is equinumerous to B .

Prop. 1 Let A , B , & C be sets. Then

- (a) $A \approx A$
- (b) $A \approx B \Rightarrow B \approx A$
- (c) $(A \approx B) \wedge (B \approx C) \Rightarrow A \approx C$

Proof: (a) $i_A: A \rightarrow A$ is a bijection. So $A \approx A$.

(b) Suppose $A \approx B$. Then we can find a bijection $f: A \rightarrow B$. Since f is a bijection, $f^{-1}: B \rightarrow A$ is also a bijection. Hence: $B \approx A$. So $A \approx B \Rightarrow B \approx A$.

(c) Suppose $A \approx B$ & $B \approx C$. Then we can find bijections $f: A \rightarrow B$ & $g: B \rightarrow C$. Let $h = g \circ f$. Then $h: A \rightarrow C$ is a bijection (verify this). So $A \approx C$. $\therefore (A \approx B) \wedge (B \approx C) \Rightarrow (A \approx C)$.

Prop 2 Let A, B, C, D be sets with $A \approx B$ & $C \approx D$. Then (2)

$$(a) (A \cap C = \emptyset \wedge B \cap D = \emptyset) \Rightarrow (A \cup C \approx B \cup D).$$

$$(b) A \times C \approx B \times D$$

Proof: Since $A \approx B$ & $C \approx D$, we can find bijections $f: A \rightarrow B$ & $g: C \rightarrow D$

$$(a) \text{ Now suppose } A \cap C = \emptyset \text{ & } B \cap D = \emptyset.$$

Let $h: A \cup C \rightarrow B \cup D$ be defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in C. \end{cases}$$

Since $A \cap C = \emptyset$ & $B \cap D = \emptyset$, it follows that $h: A \cup C \rightarrow B \cup D$ is a bijection (verify this). So $A \cup C \approx B \cup D$. Hence

$$(A \cap C = \emptyset \wedge B \cap D = \emptyset) \Rightarrow (A \cup C \approx B \cup D).$$

(b) Let $h: A \times C \rightarrow B \times D$ be defined by

$$h((a, c)) = (f(a), g(c)).$$

Then $h: A \times C \rightarrow B \times D$ is a bijection.

Let us verify this. Suppose $h((a_1, c_1)) = h((a_2, c_2))$. Then $(f(a_1), g(c_1)) = (f(a_2), g(c_2))$.

So $f(a_1) = f(a_2)$ & $g(c_1) = g(c_2)$. Since f & g are bijections it follows that $a_1 = a_2$ & $c_1 = c_2$.

So $(a_1, c_1) = (a_2, c_2)$. Thus h is injective.

Now let (b, d) be any element of $B \times D$. Then

$f^{-1}(b)$ & $g^{-1}(d)$ are well-defined because f & g are bijections. So

$$\begin{aligned} h(f^{-1}(b), g^{-1}(d)) &= (f(f^{-1}(b)), g(g^{-1}(d))) \\ &= (b, d). \end{aligned}$$

Hence h is surjective. Thus $h: A \times C \rightarrow B \times D$ is a bijection. So $A \times C \approx B \times D$.

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Def

The set A is finite if we can find a natural number k such that $A \approx N_k$ where $N_k = \{0, 1, 2, \dots, k-1\} = \{n \in \mathbb{N} : n < k\}$. The set A is infinite if it is not finite.

Ex. II (a) The set $N_k = \{0, 1, 2, \dots, k-1\}$ is finite because $N_k \approx N_k$. In particular $N_0 = \emptyset$ is finite and, ^{so} also is $N_1 = \{0\}$.

(b) The set $\{2, 5, 7\}$ is finite because $\{2, 5, 7\} \approx N_3$.

Prop. 3 Let A & B be finite sets and $C \subseteq A$. Then

- (a) C is finite
- (b) $A \cap B$ is finite
- (c) $A \cup B$ is finite
- (d) $A \times B$ is finite.

Proof: Since A & B are finite we can find bijections $f: A \rightarrow N_k$ and $g: B \rightarrow N_l$.

(a) If $C = \emptyset$, there is nothing to prove because \emptyset is already finite. So suppose $C \neq \emptyset$. Let $C_0 = C$ and define $x_0 = \text{smallest element of } f[C_0]$. In general, given that C_i & x_i have been defined, define $C_{i+1} = C_i - \{x_i\}$ and $x_{i+1} = \text{smallest element of } f[C_{i+1}]$. Then for some t we must have $C_t = \emptyset$ because $f[C_0] \subseteq f[A]$ which has only k elements. Let $n = \text{smallest } i$ such that $f[C_i] = \emptyset$. Then $C = \{x_0, \dots, x_{n-1}\}$ and $h: C \rightarrow N_n$ with $h(x_i) = i$ is a bijection. So $C \approx N_n$ and is thus finite.

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(b) $A \cap B$ is a subset of A , so $A \cap B$ is finite by part (a).

(c) $A \cup B = (A - B) \cup B$. Since $A - B$ is a subset of A , it is finite. So we can find a bijection $f_1: A - B \rightarrow \mathbb{N}_n$ for some $n \in \mathbb{N}$. Let $h: A \cup B \rightarrow \mathbb{N}_{n+l}$ be defined by

$$h(x) = \begin{cases} f_1(x) & \text{if } x \in A - B, \\ n + g(x) & \text{if } x \in B. \end{cases}$$

Then h is a bijection. So $A \cup B$ is finite.

(d) If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$ & so is finite.

Now suppose $A \neq \emptyset$ & $B \neq \emptyset$. Then $k \geq 1$ & $l \geq 1$.

Define $h: A \times B \rightarrow \mathbb{N}_{k+l}$ by

$$h((a, b)) = f(a) + k \cdot g(b)$$

Then h is a bijection. So $A \times B$ is finite.

Let us verify that h is a bijection. Suppose

$$h((a_1, b_1)) = h((a_2, b_2)). \text{ Then}$$

$$f(a_1) + k \cdot g(b_1) = f(a_2) + k \cdot g(b_2)$$

Since $f(a_1)$ & $f(a_2)$ are in $\{0, 1, \dots, k-1\}$,

it follows that $f(a_1) = f(a_2)$. So $a_1 = a_2$

because f was a bijection. Thus $k \cdot g(b_1)$

$= k \cdot g(b_2)$. Since $k \neq 0$, $g(b_1) = g(b_2)$ and so $b_1 = b_2$ bec. g is bijective. So h is injective.

Now let n be any element of $\{0, 1, 2, \dots, k \cdot l - 1\}$

Then we can express n in the form

$$n = n_1 + k \cdot n_2 \quad \text{where } 0 \leq n_1, n_2 \leq l-1.$$

Put $a = f^{-1}(n_1)$ & $b = g^{-1}(n_2)$. Then

$$\begin{aligned} h((a, b)) &= f(f^{-1}(n_1)) + k \cdot g(g^{-1}(n_2)) \\ &= n_1 + k \cdot n_2 = n. \end{aligned}$$

So h is surjective. Hence h is bijective.]

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Prop. 4 (a) \mathbb{N} is an infinite set

- (b) If A is infinite & $A \subseteq B$, then B is infinite.
- (c) If $A \approx \mathbb{N}$, then A is infinite.

Proof: (a) Suppose \mathbb{N} was finite. Then we can find a $k \in \mathbb{N}$ such that $\mathbb{N} \approx \mathbb{N}_k$. Since $\mathbb{N} \approx \mathbb{N}_k$, $\mathbb{N}_k \approx \mathbb{N}$. So we can find a bijection

$f: \mathbb{N}_k \rightarrow \mathbb{N}$. Let $n = \max \{f(0), \dots, f(k-1)\} + 1$.

Then $n > f(i)$ for each $i \in \mathbb{N}_k$. But $n \in \mathbb{N}$ and since $n > f(i)$ for each $i \in \mathbb{N}_k$, $n \notin f[\mathbb{N}_k]$. Hence f is not surjective. But this contradicts the fact that f was a bijection. Hence \mathbb{N} cannot be finite. So \mathbb{N} is infinite.

(b) Suppose A is infinite & $A \subseteq B$. Now if B was finite, then A would be a subset of a finite set, and so A would be finite. But this would contradict the fact that A was infinite. Hence B cannot be finite.

Thus B must be infinite. So if A is infinite & $A \subseteq B$, then B is infinite.

(c) Suppose $A \approx \mathbb{N}$. Then $\mathbb{N} \approx A$. Now if A was finite, then we would be able to find a $k \in \mathbb{N}$ such that $A \approx \mathbb{N}_k$. Since $\mathbb{N} \approx A$ & $A \approx \mathbb{N}_k$, it follows that $\mathbb{N} \approx \mathbb{N}_k$ which means that \mathbb{N} would be finite - a contradiction. Hence A cannot be finite. So A is infinite.

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§2. Denumerable & Countable sets.

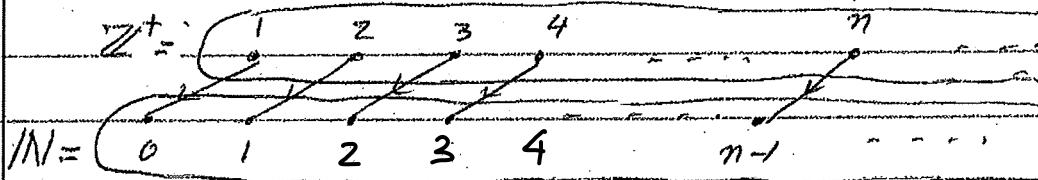
Def. The set A is denumerable if $A \approx N$. The set A is countable if A is finite or denumerable.

Prop. 5 The following sets are denumerable

- (a) \mathbb{Z}^+ (b) \mathbb{Z} (c) $2\mathbb{Z}$ (d) $N \times N$

Proof: (a) Let $f: \mathbb{Z}^+ \rightarrow N$ be defined by $f(k) = k-1$.

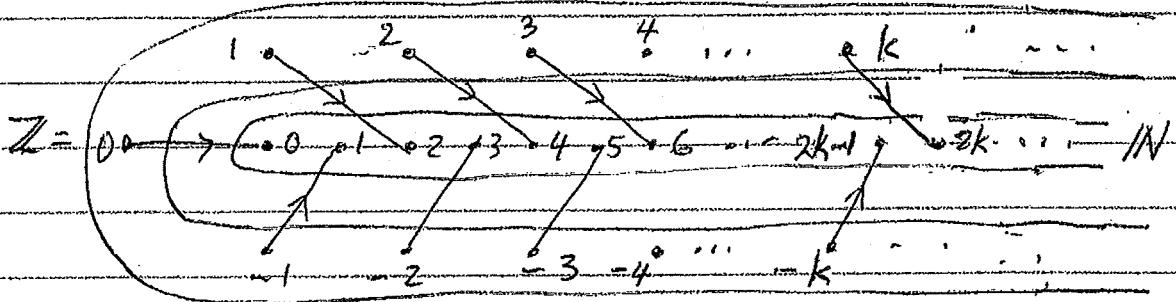
Then f is a bijection. So $\mathbb{Z}^+ \approx N$ & hence \mathbb{Z}^+ is denumerable



(b) Let $f: \mathbb{Z} \rightarrow N$ be defined by $f(k) = \begin{cases} 2k & \text{if } k \geq 0, \\ -(2k+1) & \text{if } k < 0. \end{cases}$

Then f is a bijection.

So $\mathbb{Z} \approx N$ & hence \mathbb{Z} is denumerable.



(c) Let $f: 2\mathbb{Z} = \{2k: k \in \mathbb{Z}\} \rightarrow N$ be defined by

$$f(l) = \begin{cases} l & \text{if } l \geq 0 \\ -(l+1) & \text{if } l < 0 \end{cases}$$

Then f is a bijection. So $2\mathbb{Z} \approx N$. So $2\mathbb{Z}$ is denumerable.

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(d) Let $f: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$f((k, l)) = 2^k \cdot (2l+1) - 1.$$

Then f is a bijection. So $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ and hence $\mathbb{N} \times \mathbb{N}$ is denumerable.

$$(0, 3) \quad (1, 3) \quad (2, 3) \quad (3, 3) \quad 2^0 \cdot 7 - 1 \quad 2^1 \cdot 7 - 1 \quad 2^2 \cdot 7 - 1$$

$$(0, 2) \quad (1, 2) \quad (2, 2) \quad (3, 2) \quad 2^0 \cdot 5 - 1 \quad 2^1 \cdot 5 - 1 \quad 2^2 \cdot 5 - 1$$

$$(0, 1) \quad (1, 1) \quad (2, 1) \quad (3, 1) \quad 2^0 \cdot 3 - 1 \quad 2^1 \cdot 3 - 1 \quad 2^2 \cdot 3 - 1 \quad 2^3 \cdot 3 - 1$$

$$(0, 0) \quad (1, 0) \quad (2, 0) \quad (3, 0) \quad 2^0 \cdot 1 - 1 \quad 2^1 \cdot 1 - 1 \quad 2^2 \cdot 1 - 1 \quad 2^3 \cdot 1 - 1$$

 $\mathbb{N} \times \mathbb{N}$ \mathbb{N}

Let us really check that f is indeed a bijection.

Suppose $f((k_1, l_1)) = f((k_2, l_2))$. Then

$$2^{k_1} \cdot (2l_1 + 1) - 1 = 2^{k_2} \cdot (2l_2 + 1) - 1$$

So $2^{k_1} \cdot (2l_1 + 1) = 2^{k_2} \cdot (2l_2 + 1)$. Since $(2l_1 + 1)$ & $2(l_2 + 1)$ are both odd, we must have $2^{k_1} = 2^{k_2}$.

So $k_1 = k_2$. Thus $2l_1 + 1 = 2l_2 + 1$. Hence

$2l_1 = 2l_2$ & so $l_1 = l_2$. Thus $(k_1, l_1) = (k_2, l_2)$.

Hence f is injective.

Now let b be any element of \mathbb{N} . Then $b+1 \in \mathbb{N}$ and we can express $b+1$ in the form $2^k \cdot (2l+1)$ where $k, l \in \mathbb{N}$. So

$$f((k, l)) = 2^k \cdot (2l+1) - 1 = (b+1) - 1 = b$$

Hence f is surjective. Thus f is a bijection.

H.W. Show $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$, $f(k, l) = 2^{k-1} \cdot (2l-1)$ is a bijection.

Theorem 6: Suppose $A \neq \emptyset$. Then the following are equivalent

- (a) A is countable
- (b) There exists a surjection $f: \mathbb{N} \rightarrow A$
- (c) There exists an injection $g: A \rightarrow \mathbb{N}$

Proof: (a) \Rightarrow (b): Suppose A is countable. Then A is finite or A is denumerable. Now if A is denumerable, then $A \approx \mathbb{N}$. So $\mathbb{N} \approx A$. Hence we can find a bijection $f: \mathbb{N} \rightarrow A$. Since f is a bijection, it is a surjection. And if A is finite, then $A \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$. So $\mathbb{N}_k \approx A$. Hence we can find a bijection $h: \mathbb{N}_k \rightarrow A$. Now define $f: \mathbb{N} \rightarrow A$ by

$$f(x) = \begin{cases} h(x) & \text{if } x \in \mathbb{N}_k \\ h(0) & \text{if } x \in \mathbb{N} - \mathbb{N}_k \end{cases}$$

Then f is a surjection because h was surjective. So in either case we get a surjection $f: \mathbb{N} \rightarrow A$.

(b) \Rightarrow (c): Suppose there is a surjection $f: \mathbb{N} \rightarrow A$. Then $f^{-1}[\{a\}] = \{n \in \mathbb{N} : f(n) = a\} \neq \emptyset$ for each $a \in A$. Define $g: A \rightarrow \mathbb{N}$ by $g(a) = \text{smallest element of } f^{-1}[\{a\}]$. Then $f \circ g = i_A$. Since f is surjective it follows that g is injective.

(c) \Rightarrow (a): Suppose there is an injective function $g: A \rightarrow \mathbb{N}$. Let $B = \text{ran}(g)$. Then $g: A \rightarrow B$ is a bijection. So $A \approx B$. We will show that B is finite or denumerable. Suppose B is not finite. We shall show that B is denumerable. From this it will follow that B is finite or denumerable. So B , & hence A , will be countable.

(c) \Rightarrow (a): Define $h: \mathbb{N} \rightarrow B$ by $h(n) = (n+1)$ -th smallest element of B . Since B is not finite, $h(n)$ will exist for each $x \in B$. So h is well-defined. Also h is injective by its definition. And finally since $B \subseteq \mathbb{N}$, $\{h(n) : n \in \mathbb{N}\}$ must exhaust B , because the elements of B can be well-ordered. Hence $\text{ran}(h) = B$. So h is a bijection. Thus $\mathbb{N} \approx B$. So $B \approx \mathbb{N}$. Hence B is denumerable.

Prop. 7 Suppose A & B are non-empty countable sets. Then
 (a) $A \cup B$ is countable
 (b) $A \times B$ is countable

Proof: Suppose A & B are non-empty countable sets. Then we can find injective functions $f: A \rightarrow \mathbb{N}$ and $g: B \rightarrow \mathbb{N}$, by Theorem 6.
 (a) Let $h: A \cup B \rightarrow \mathbb{N}$ be defined by $h(x) = \begin{cases} 2f(x) & \text{if } x \in A \\ 2g(x) + 1 & \text{if } x \in B \end{cases}$.

Then h is injective. So $A \cup B$ is countable by Theorem 6.

(b) Let $h: A \times B \rightarrow \mathbb{N}$ be defined by

$$h((a, b)) = 2^{f(a)} \cdot [2g(b) + 1]$$

Then h is an injective function. So $A \times B$ is countable by Theorem 6.

Theorem 8: Let $\langle A_i : i \in I \rangle$ be an indexed family of non-empty countable sets. and suppose I is also non-empty & countable. Then $\bigcup_{i \in I} A_i$ is countable.

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Proof: Suppose I is non-empty & countable. Then we can find an surjection $g: \mathbb{N} \rightarrow I$. Also since each A_i is non-empty & countable, we can find surjections $f_i: \mathbb{N} \rightarrow A_i$ for each $i \in I$. Let $h: \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_{i \in I} A_i$ be defined by $h((k, l)) = f_{g(k)}(l)$.

Then h will be a surjection. Now let $j: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be the bijection defined by $j((k, l)) = 2^k(2l+1) - 1$. Then $j^{-1}: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ will also be a bijection and $h \circ j^{-1}: \mathbb{N} \rightarrow \bigcup_{i \in I} A_i$ will be a surjection. So by Theorem 6, $\bigcup_{i \in I} A_i$ will be countable.

Prop. 9: \mathbb{Q} is denumerable.

Proof: We know that $\mathbb{Q} = \bigcup_{n \in \mathbb{Z}^+} \left(\frac{1}{n} \mathbb{Z} \right)$. Now $\frac{1}{n} \mathbb{Z} = \left\{ \frac{k}{n} : k \in \mathbb{Z} \right\}$ is denumerable because \mathbb{Z} is denumerable. Also \mathbb{Z}^+ is denumerable. So \mathbb{Q} is a countable union of countable sets. By Theorem 8, it follows that \mathbb{Q} is countable. Since $\mathbb{N} \subseteq \mathbb{Q}$, \mathbb{Q} is not finite. Hence \mathbb{Q} must be denumerable.

Prop. 10: \mathbb{N}^* is denumerable.

Proof: $\mathbb{N}^* = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$. Since \mathbb{N} is denumerable and \mathbb{N}^k is denumerable for all $k \in \mathbb{Z}^+$ & $\mathbb{N}^0 = \{<\}\$, it follows that \mathbb{N}^* is countable, by Theorem 8. Since $\mathbb{N}^1 \subseteq \mathbb{N}^*$ & \mathbb{N}^1 is denumerable, \mathbb{N}^* is not finite. So \mathbb{N}^* is denumerable.

§3. Uncountable sets.

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Def. A set A is uncountable if it is not countable.

Theorem 10 (Cantor's Diagonal Power-set Theorem for \mathbb{N})
 $P(\mathbb{N})$ is uncountable.

Proof: Let $f: \mathbb{N} \rightarrow P(\mathbb{N})$ be any function. We will show that f is not surjective by finding a set $D \in P(\mathbb{N})$ such that $D \notin \text{ran}(f)$. Since this will be true for any f , it will follow that there is no surjective function from \mathbb{N} to $P(\mathbb{N})$. So $P(\mathbb{N})$ will not be countable by Theorem 8. Hence $P(\mathbb{N})$ will be uncountable.

Let $D = \{n \in \mathbb{N} : n \notin f(n)\}$. Then $D \subseteq \mathbb{N}$. So $D \in P(\mathbb{N})$. Let us suppose $D \in \text{ran}(f)$. Then we can find an $n_0 \in \mathbb{N}$ such that $f(n_0) = D$.

Now since $D \subseteq \mathbb{N}$, either $n_0 \in D$ or $n_0 \notin D$. But if $n_0 \in D$, then $n_0 \notin f(n_0) = D$ — which is a contradiction. And if $n_0 \notin D$, then $n_0 \in f(n_0) = D$ by the definition of D — which is also a contradiction. So in either case we got a contradiction.

Theorem 11: \mathbb{R} is uncountable.

Proof: We will find an injection $f: P(\mathbb{N}) \rightarrow \mathbb{R}$. Now if \mathbb{R} was countable, then we can find an injection $g: \mathbb{R} \rightarrow \mathbb{N}$. But then $g \circ f: P(\mathbb{N}) \rightarrow \mathbb{N}$ would be

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an injection - and thus $P(N)$ would be countable. But this would contradict Theorem 10. Hence \mathbb{R} cannot be countable. So \mathbb{R} must be uncountable.

Let us now define our injection $f: P(N) \rightarrow \mathbb{R}$. Take any $A \in P(N)$. Then $A \subseteq N$. Put

$$d_n^A = \begin{cases} 3 & \text{if } n \in A \\ 7 & \text{if } n \notin A \end{cases}$$

and let $f(A) = d_0^A.d_1^A.d_2^A.d_3^A \dots d_n^A \dots$. We will show that f is injective. Suppose $A, B \in P(N)$ and $A \neq B$. Then $(A-B) \cup (B-A) \neq \emptyset$. So we can find a $k \in (A-B) \cup (B-A)$. Now

If $k \in A-B$, then $d_k^A = 3$ & $d_k^B = 7$.

And if $k \in B-A$, then $d_k^A = 7$ & $d_k^B = 3$.

So in either case $d_k^A \neq d_k^B$. So $f(A) \neq f(B)$ because they will differ in the k -th decimal place by 4. Hence f is injective & we are done.

Def. We say that A is inferior to B , and write $A \preceq B$, if we can find an injection $g: A \rightarrow B$.
We say that A is superior to B , and write $A \succeq B$ if we can find a surjection $f: A \rightarrow B$.

Theorem 12 (Cantor-Schröder-Bernstein Theorem)

Let A & B be sets. If $A \preceq B$ and $B \preceq A$, then $A \approx B$.

Proof: See textbook.

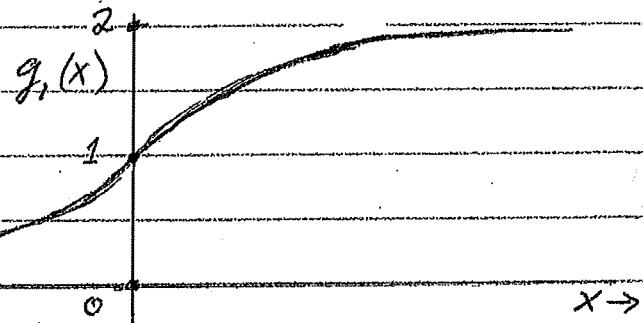
Note: We can also rephrase this theorem as saying
If $f: A \rightarrow B$ & $g: B \rightarrow A$ are injections, then \exists a bijection $h: A \rightarrow B$.

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Theorem 13: $\mathbb{R} \approx P(\mathbb{N})$.

Proof: In Theorem 12, we found an injection $f: P(\mathbb{N}) \rightarrow \mathbb{R}$. So $P(\mathbb{N}) \preceq \mathbb{R}$. We will now find an injection $h: \mathbb{R} \rightarrow P(\mathbb{N})$. From this it will follow that $\mathbb{R} \preceq P(\mathbb{N})$ and from Theorem 12, we will get that $\mathbb{R} \approx P(\mathbb{N})$.

First observe that $g_1: (0, 1) \rightarrow \mathbb{R}$ with $g_1(x) = \tan^{-1} \left[\frac{\pi}{2}(x-1) \right] + 1$ is a bijection from $(0, 1)$ to \mathbb{R} . So $(0, 1) \approx \mathbb{R}$.



Now define $g_2: (0, 1) \rightarrow P(\mathbb{N})$ as follows. Express each $x \in (0, 1)$ as a unique infinite binary-decimal which never terminates in an infinite sequence of 1's. So $x = a_0.a_1a_2a_3\dots a_n\dots$ with $a_i \in \{0, 1\}$.

Put $A_x = \{n \in \mathbb{N} : a_n \neq 0\}$. Then $A_x \in P(\mathbb{N})$.

Let $g_2(x) = A_x$. Then g_2 will be injective because of its definition. Let $h = g_2 \circ g_1^{-1}$.

Then $h: \mathbb{R} \rightarrow P(\mathbb{R})$ will be an injective function because g_2 is injective and g_1^{-1} is bijective. Hence we got our injective function and we are done.

END.